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TOPICS IN THE THEORY OF ANALYTIC FUNCTIONS (III)*

Integral Functions; Picard's theorem and its extensions; Meromorphic Functions; Conformal Representation; Bloch's constant; Schlicht Functions.

BY

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1. An integral function has been defined [I, §8·3] as a uniform analytic function regular in the whole finite plane. If $z=\infty$ be a pole of order m for an integral function $f(z)$, then $f(z)$ is a polynomial of degree m. If $z=\infty$ is an essential singularity, then $f(z)$ is called a *transcendental integral function.* An integral function can be expressed as a power-series

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \qquad \ldots \quad (1)$$

which converges for all finite values of z.

1·1. We have stated [I, §8·3] that the only uniform analytic function without any singularity is a constant. There is an important extension of this result due to Liouville. *If $f(z)$ is an integral function such that $|f(z)| \leqslant A$ (A being a finite positive number), then $f(z)$ is a constant.* This is an easy deduction from a result due to Weierstress *that if $z=a$ be an isolated essential singularity of a function $f(z)$ and λ any number, finite or infinite, there exists a sequence $\{z_n\}$ such that $z_n \to a$ and $f(z_n) \to \lambda$ as $n \to \infty$.*

1·2. Picard's Theorem†

The fundamental theorem of Algebra states that a *polynomial of degree m takes any given value exactly at m points.* If we regard

* The first article of this series appeared in *The Mathematics Student*, Vol. 5, pp. 1-24 and the second, in Vol. 7, pp. 16-24. These will be referred to as (I) and (II) respectively. The notations and definitions of these articles are followed throughout this paper.

† E. C. Titchmarsh, *Theory of Functions*, pp. 282-283.

integral functions as polynomials of infinite degree, we should expect that an integral function would take any value an infinite number of times. This is, in fact, true with one modification, namely, that there is the possibility of the existence of one value for which the above statement would not be true. This is the content of Picard's famous theorem. Stated in a slightly more general form, Picard's theorem runs as follows: *If $z=a$ be an isolated essential singularity of $f(z)$, then $f(z)$ takes any given value, with one possible exception, an infinity of times in any neighbourhood $c(a, < r)$ of $z=a$.* That an exceptional value might exist is seen by taking $f(z)=e^z$ which does not assume the value zero at all for any finite z. But $e^z \longrightarrow 0$ as $z \longrightarrow \infty$ along the negative real axis (cf. Weierstras's result stated at the end of § 1·1)

2. Integral functions of finite order.

Let $f(z)$ be an integral function and let

$$M(r, f) = \max_{|z| \leq r} |f(z)| \qquad \ldots \quad (2)$$

By the maximum Modulus principle (II § 4 and §4·1), $M(r, f)$ is a steadily increasing function of r. By Liouville's theorem (§ 1·2) $M(r, f) \longrightarrow \infty$ unless $f(z)$ is a constant. As a generalisation of this result we can state that if

$$M(r, f) \leqslant Ar^k, \qquad \ldots \quad (3)$$

for some finite k, then $f(z)$ is a polynomial of degree not exceeding k. Hence for a transcendental integral function $M(r, f)$ increases more rapidly than any power of r however large. The next simplest case is when $M(r, f)$ is comparable with a function of the form e^{r^ρ}, $\rho > 0$. If there be a finite ρ such that

$$\log M(r, f) \leqslant Ar^\rho \qquad \ldots \quad (4)$$

then $f(z)$ is called a *function of finite order.* The precise order of $f(z)$ is defined as the lower bound of values of ρ for which (4) holds and is given by the expression

$$\overline{\lim_{r\to\infty}} \frac{\log \log M(r, f)}{\log r} \qquad \ldots \quad (5)$$

Functions of finite order have been the most widely studied among integral functions. Their properties have beeen exhaustively investigated and a rich collection of results have been obtained for which the reader is referred to relevant books and memoirs.*

* G. VALIRON, *General Theory of Integral Functions* and the references given therein.

2·1. Extensions of Picard's theorem.

For integral functions of finite order, Picard's theorem has been made more precise. Let $f(z)$ be an integral function of order ρ. Let $\{z_n(a)\}$ be the zeros of $f(z)-a$. The lower bound $\mathrm{E}(a)$ of values of h such that

$$\sum |z_n(a)|^{-h} \qquad \ldots \quad (6)$$

converges, is called the exponent of convergence of the sequence $\{z_n(a)\}$. It is easily proved that $\mathrm{E}(a) \leqslant \rho$. But Borel has proved that

$$\mathrm{E}(a)=\rho, \qquad \ldots \quad (7)$$

for all a except possibly one for which $\mathrm{E}(a)$ *might be less than* ρ. This is the preciser form of Picard's theorem for functions of finite order. Various extensions of these results are known all of which are included in a general theorem for meromorphic functions to be stated later. (see, § 4·1).

2·2. Integral functions with the given zeros.

Given the points where a polynomial vanishes, that is, give its zeros, we can always construct the polynomial but for a constant factor. The corresponding problem in the theory of integral functions is to construct an integral function with given zeros and to find all integral functions having only these zeros. Let

$$\mathrm{E}(u, p)=(1-u)\, e^{u+\frac{u^2}{2}+\cdots+\frac{u^p}{p}} \qquad \ldots \quad (7)$$

where p is a positive integer. Let $\{z_n\}$ be the points where the integral function is to vanish. We suppose that $z_n \neq 0$. If the required function has a zero at $z=0$ let its order be k. Then a result*, *known as the factorization theorem of Weierstrass states that a sequence of integers* $\{p_n\}$ *depending on the sequence* $\{z_n\}$, $(|z_n| \to \infty)$ *exists such that the product*

$$h(z)=z^k \prod_{n=1}^{\infty} \mathrm{E}\left(\frac{z}{z_n}, p_n\right) \qquad \ldots \quad (8)$$

converges for all z and is an integral function with the required zeros. Moreover, any $f(z)$ *with these zeros is of the form*

$$f(z) = e^{g(z)}\, h(z)\,, \qquad \ldots \quad (9)$$

where $g(z)$ *is an integral function.*

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* E. C. Titchmarsh, l.c., pp. 246-284. When the function is of finite order we can take each $p_n=p$ where p is the least integer such that $\Sigma|z_n|^{-(p+1)}$ converges. It can be shown that $p \leqslant \rho \leqslant p+1$ where ρ is the order of $f(z)$.

2·3. Integral functions with assigned values at a sequence of points.

Let $\{z_n\}$ and $\{y_n\}$ be two sequence of values and let z_n be distinct, and $|z_n| \to \infty$ as $n \to \infty$. Does there exist an integral function $f(z)$ such that $f(z_n) = y_n$? The answer is in the affirmative. The general expression for such an integral function can be obtained as follows. Let $h(z)$ be an integral function with simple zeros at $\{z_n\}$ constructed as in § 2·2. Let $f(z)$ be any function such that $f(z_n) = y_n$. *Then there exists a sequence of integers* $\{q_n\}$ *such that*

$$\sum_{n=1}^{\infty} \frac{y_n}{h'(z_n)} \frac{1}{z - z_n} \left(\frac{z}{z_n}\right)^{q_n} \quad \ldots \quad (10)$$

converges for all z *except at* $\{z_n\}$ *and*

$$f(z) = h(z) \left\{ g(z) + \sum_{n=1}^{\infty} \frac{y_n}{h'(z_n)} \frac{1}{z - z_n} \left(\frac{z}{z_n}\right)^{q_n} \right\}, \quad \ldots \quad (11)$$

where $g(z)$ *is any integral function.* This result also is due to Weierstrass.

3. Meromorphic functions.

A uniform function regular except for poles *in the finite plane* is called a *meromorphic function.* A meromorphic function without any poles is an integral function. At a pole of order m, the function $f(z)$ is supposed to take the value ∞ m times [I, § 5·1]. We have stated that a function having only a finite number of poles in the *whole complex plane* is a rational function [I, § 8·3]. A meromorphic function which is not a rational function will have $z = \infty$ as an essential singularity. Any meromorphic function $f(z)$ can be expressed in the form

$$f(z) = g(z)/h(z) \quad \ldots \quad (12)$$

where $g(z)$ is an integral function whose zeros are those of $f(z)$ and $h(z)$ is an integral function whose zeros are the poles of $f(z)$.

3·1. Picard's theorem for meromorphic functions :

For meromorphic functions Picard's theorem can be stated as follows: *A meromorphic function takes any value (finite or infinite), with the possible exception of at most two values, an infinite number of times unless it reduces to a rational function.* It may be noted that an integral function is a meromorphic function which does not take the value ∞. Moreover if a meromorphic function $f(z)$ does not take the value α then $1/\{f(z) - \alpha\}$ is an integral function, so that Picard's theorem for meromorphic and integral functions can be deduced one from the other.

3·2. Jensen's Formula.

Before proceeding to define meromorphic functions of finite order we shall state an important result known as Jensen's formula. Let $f(z)$ be a function meromorphic in the circle $|z| \leqslant r$. Let* $n(r, a)$ denote the number of zeros of $f(z) - a = 0$ in $|z| \leqslant r$. Let

$$N(r, a) = \int_0^r \frac{n(t, a) - n(0, a)}{t} dt + n(0, a) \log r; \qquad \ldots \quad (13)$$

then

$$N(r, 0) - N(r, \infty) = \frac{1}{2\pi}\int_0^{2\pi} \log |F(re^{i\theta})| \, d\theta - \log \lambda, \qquad \ldots \quad (14)$$

where † *is a constant the precise value of which is given by*

$$\lambda = \lim_{z \to 0} |z^{-k} f(z)|$$

where † $k = n(0, 0)$ *if* $z = 0$ *is a zero of* $f(z)$ *and* $k = -n(0, \infty)$ *if* $z = 0$ *is a pole of* (z) *and otherwise* $k = 0$

4. Meromorphic functions of finite order.

The function $M(r, f)$ defined by (2) is no longer an increasing function of r when $f(z)$ is a meromorphic function. We shall introduce another function which will measure the behaviour of $f(z)$ as $|z| \to \infty$. Let $\log^+ \lambda = \max(0, \log \lambda)$ and let

$$m(r, a) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a|} d\theta \qquad \ldots \quad (14)$$

where $1 / \{f(z) - a\}$ is to be replaced by $f(z)$ when $a = \infty$.

We put

$$T(r, f) = m(r, \infty) + N(r, \infty) \qquad \ldots \quad (15)$$

where $N(r, \infty)$ is defined by (13). Nevanlinna ‡ has shown that $T(r, f)$ is an increasing convex function of $\log r$ (II § 4·1) and that if

$$T(r, f) \leqslant A \log r,$$

then $f(z)$ is a rational function. We use $T(r)$ and define the order ρ of $f(z)$ by the relation

$$\rho = \overline{\lim_{r \to \infty}} \frac{\log T(r, f)}{\log r}. \qquad \ldots \quad (16)$$

When $f(z)$ is an integral function it can be shown that

$$T(r, f) \leqslant \log M(r, f) \leqslant \frac{k+1}{k-1} T(kr, f), \qquad \ldots \quad (17)$$

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* $n(r, \infty)$ means the number of poles in $|z| \leqslant r$. $n(0, a)$ means the order of the zero of $f(z) - a = 0$ at $z = 0$.

† k is the order of zero or the negative of the order of the pole at $z = 0$ as the case may be.

‡ R. Nevanlinna, *Fonctions Meromorphes*, (Paris, 1923).

for any $k > 1$ so that the definitions (5) and (16) gives the same value for the order of $f(z)$ when it is an integral function. When $f(z)$ is of finite order ρ, the functions $g(z)$ and $h(z)$ in (12) can be chosen to be of order not exceeding ρ while one at least is of order ρ.

4·1. Picard's theorem has been considerably extended by Nevanlinna*. His main result can be stated as follows: *Let $f(z)$ be a meromorphic function of finite order ρ and let a, b, c be three distinct complex numbers. Then*

$$T(r, f) \leqslant N(r, a) + N(r, b) + N(r, c) + O(\log r)$$

where $O(\log r)$ means a quantity less than $A \log r$. ... (18)

From (18) we can deduce that $E(a)$ defined in § 2·1 satisfies the relations $E(a) \leqslant \rho$ for all a while $E(a) = \rho$ except possibly for two values of a at most. There is a corresponding extension of (18) for any meromorphic function.

5. Conformal representation.

Let

$$w = f(z) \qquad \text{... (19)}$$

be a single valued function regular in a domain D. One of the central problems in the theory of analytic functions is to study the picture of D in the w-plane given by the relation (19). Let $w = w_0$ be a point corresponding to $z = z_0$ given by (19) where $f'(z_0) \neq 0$. Let c_1 and c_2 be two Jordan arcs passing through $z = z_0$ in the z—plane and having definite tangents at $z = z_0$. Let γ_1 and γ_2 be the corresponding curves in the w-plane passing through $w = w_0$. *Then the existence of $f'(z_0) \neq 0$ implies that the angle between the tangents at $z = z_0$ to the curves c_1 and c_2 is equal to that between the tangents at $w = w_0$ to γ_1 and γ_2 and the sense of rotation from c_1 to c_2 is the same as that from γ_1 to γ_2.* That is, small corresponding figures in the z-and w-plane are approximately similar. Therefore the representation given by (19) is called *conformal.* Since the points $f'(z) = 0$ are isolated, conformality in the representation (19) is preserved except at these isolated points. A modified statement† can be made at points where $f'(z) = 0$.

5·1. If R is the point set corresponding to D given by (19), it can be shown that R is a domain in the w-plane. But the correspondence given by (19) need not be one to one. When the correspondence is one to one, *the domain D is said to be bi-uniformly*

* R. Nevanlinna, *Fonctions Meromorphes*, (Paris, 1923).

† C. Caratheodory, *Conformal Representation*, (Cambridge Tracts, No. 28.) and E. C. Tichmarsh, l.c., p. 189.

mapped on the domain R. Usually when we speak of conformal representation we mean bi-uniform mapping. We shall adhere to this convention unless otherwise stated.

5·2. The *connectivity* of a domain has been already defined [I, § 1·7]. In any conformal representation of a domain D on another R, the connectivity is preserved. That is, if D is simply connected so is R; if D is doubly connected the same is true of R and so on.

6. *The Fundamental Theorem of Conformal Representation.*

The circle C $(0, < 1)$, i.e. $|z|<1$ is called the unit circle; it can be conformally represented on any other circle. The main theorem of conformal representation can be stated as follows: *Any simply connected domain D with at least two boundary points can be conformally represented on the unit circle.* A simply connected domain D with only one boundary point can be represented on the finite plane (that is, the whole complex plane with the exclusion of $z=\infty$).

6·1. The bilinear transformation

$$w=\frac{az+b}{cz+d},\; ad-bc\neq 0, \qquad \ldots \quad (20)$$

transforms the complex plane into itself conformally. Conversely, a conformal transformation of the plane into itself is given by a relation of the form (20). The transformation (20) can be visualized geometrically as a succession of an even number of inversions in circles suitably chosen. It transforms the circumference of a circle into that of another circle, the interior of the first circle going into the interior or the exterior of the other circle. The most general conformal transformation which transforms the interior of the unit circle into itself is given by

$$w=\alpha\,(z-z_0)/(1-z\bar{z}_0),\;\; |z_0|<1, \qquad \ldots \quad (21)$$

where $|\alpha|=1$ and conversely,

6·2. Schwaz's lemma: *Let $f(z)$ be regular in $C\,(0, <1)$ and let $|f(z)|\leqslant M$ in $|z|<1$ and $f(0)=0$. Then*

$$|f(z)|\leqslant M\,|z|, \qquad \ldots \quad (21)$$

for $|z|<1$. Moreover, if the equality sign holds in (21) *for some one point in $C\,(0, <1)$ then it holds throughout this circle and $f(z)=Me^{i\lambda}z$ where λ is real.* This result is known as Schwaz's lemma and though it is an easy deduction from the Maximum Modulus principle (II, §4 and 4·1) its applications are very extensive and it forms one of the central theorems in the theory of Conformal Representation.

6·3. The Boundary Problem.

Let D be a simply connected domain with at least two boundary points and let $w=f(z)$ represent D conformally on the unit circle $|w|<1$ in the w-plane. One of the main problems in the Theory of Conformal Representation is the following: Does the function $w=f(z)$ set up a one to one correspondence between the boundary $B(D)$ of D and the boundary $|w|=1$ of the circle?

Suppose z_0 is a point of B (D) and let $\{z_n\}$ be any sequence in D tending to z_0. Suppose the corresponding sequence $\{w_n\}$ tends to a point w_0 on $|z|=1$. Moreover suppose that we get the same point w_0 by taking any sequence $\{z_n\}$ tending to z_0 and further that different points w_0 are obtained for different z_0 on B (D). This establishes a one to one correspondence between B (D) and $|w|=1$ and this correspondence is *entirely* determined by $w=f(z)$. But the suppositions made above do not always hold. For instance, the same z_0 might give rise to different w_0 by choosing the sequence $\{z_n\}$ properly. And several other conceivable possibilities can happen in the general case. But *if $B(D)$ consists of a simple closed Jordan curve γ, there is a one to one correspondence between γ and $|w|=1$. If $B(D)$ contains a simple Jordan arc δ, then δ corresponds to an arc of $|w|=1$.* In general, the boundary of a simply connected domain can be of very complicated type and then the statement above is not valid. For an exhaustive discussion of these questions reference is made elsewhere*. The Theory of Normal families (II, § 5·5·7) has important applications in this connection.

7. Inversion of analytic functions.

Let $w=f(z)$ be regular in a domain D. Let $z=z_0$ be a point in D and let $w_0=f(z_0)$ and $f'(z_0)\neq 0$. Then there is a circle $c\ (z_0, <p)$, $p>0$, such $f(z)$ represents this circle bi-uniformly on a certain domain R (w_0). Therefore the inverse function $z=g(w)$ is regular in R (w_0) and therefore also in any circle contained in R (w_0). This gives a proof of the statement in § 5·1 that R is a domain when D is a domain.

7·1. Let $f(z)$ be a function regular in a domain D. If $f(z)$ takes none of its values more than once in D, then *we shall say that $f(z)$ is univalent or* ***schlicht*** *in the domain D.* The result stated in the previous paragraph means that if $z=z_0$ be a point where $f'(z_0)\neq o$ then $f(z)$ is schlicht in a sufficiently small neighbourhood of z_0 and the same is

* A. L. Ford, *Automorphic Functions.*

true of the inverse function near $w_0=f(z_0)$. If $f'(z_0)=0$ then $w=f(z_0)$ will, in general, be critical points for the inverse function $z=g(w)$.

7·2 Bloch's constant:*

Let $f(z)$ be regular in $|z|<1$, $f(0)=0$, $f'(0)=1$, that is, let $f(z)$ be of the form

$$w=f(z)=z+a_2 z^2+a_3 z^3+\ldots, \qquad \ldots (22)$$

Then there is a constant $B>0$ such that for every function of the form (22), there is a circle of radius B in the w-plane which is the bi-uniform mapping of a certain domain in $|z|<1$ by the function (22). In other words, the inverse function $z=g(w)$ is schlicht in this circle of radius B. The number B is called *Bloch's constant.* The centre of the circle might vary for different functions of form (22). In §7·7·1 we have stated that if $f(z)$ is regular in a domain D then there is always a neighbourhood of $w_0=f(z_0)$ with $f'(z_0)\neq 0$ in which $z=g(w)$ is regular. The radius of the circle here will vary from point to point as well as from function to function. What Bloch's theorem states is that, for functions of form (22) regular in $|z|<1$, we can always find a circle of radius B (the same for every such function) in the w-plane where $z=g(w)$ is schlicht. For extensions and applications of this result reference is made elsewhere †

8. Schlicht functions ‡:

The properties of functions regular and schlicht in $|z|<1$ have been investigated at great length. Here again instead of considering individual schlicht functions, we consider the class of functions

$$w=f(z)=z+a_2 z^2+a_3 z^3+\cdots, \qquad \ldots (23)$$

regular and schlicht in $|z|<1$. Then it can be shown that *the picture of $|z|<1$ by (23) in the w-plane always contains the circle $|w|<\frac{1}{4}$.* It has been proved that $|a_2|\leqslant 2$ for all functions of this class and that $|f(z)|$ and $|f'(z)|$ lie between the limits

$$\left.\begin{aligned}\frac{r}{(1+r)^2} &\leqslant |f(z)| \leqslant \frac{r}{(1-r)^2}\\ \frac{1-r}{(1+r)^3} &\leqslant |f'(z)| \leqslant \frac{1+r}{(1-r)^3}\end{aligned}\right\}, \qquad \ldots (24)$$

for $|z|=r<1$. It can also be shown that

$$-\frac{4}{1-r^2}\leqslant \frac{\partial}{\partial r}\{\text{amp } f'(z)\}\leqslant \frac{4}{1-r^2}, \qquad \ldots (25)$$

* P. Dienes, *Taylor series*, (Oxford) p. 266—268.

† P. Dienes, l. c. pp. 268—272.

‡ P. Dienes, l. c. pp. 273—284.

so that taking the principal value of amp $f'(z)$ and noting that $f'(0)=1>0$ we get

$$|\text{amp } f'(z)| \leqslant 2 \log \frac{1+r}{1-r} \qquad \dots \quad (26)$$

The formula (24), (25) and (26) give limits between which the picture of $|z|=r$ by (23) lies. For several other important properties of schlicht functions reference is made elsewhere*

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* P. Dienes, l. c. and the bibliography towards the end of the book.

A NOTE ON THE CLASSIFICATION OF INTEGRAL FUNCTIONS

By

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1. Let $f(z)$ be an integral function of order $\rho > 0$, and let $M(r) = \max_{|z|=r} |f(z)|$. $f(z)$ is said to be of maximum, mean or minimum type according as $\overline{\lim_{r=\infty}} \log M(r)/r^{\rho}$ is infinite, finite and non-zero, or zero respectively. In the last case $f(z)$ is said to belong to the class of convergence or divergence according as

$$\int^{\infty} \frac{\log M(r)\, dr}{r^{\rho+1}}$$

is convergent or divergent.

Let us write $f(z) = \sum_{0}^{\infty} c_n z^n = z^k e^{g(z)} P(z)$ where $g(z)$ is a polynomial of degree q and $P(z) = \prod_{1}^{\infty} E\left(\frac{z}{a_n}, p\right)$ is canonical product of order ρ_1 and genus p formed with zeros $a_n (a_n \neq 0)$ of $f(z)$. A set of conditions involving c_n and another set of conditions involving a_n, in order that $f(z)$ may belong to a particular type, are given by Pringsheim [1,722—811].

It is known that $\rho_1 - 1 \leqslant p \leqslant \rho_1 \leqslant \rho$; $q \leqslant \rho$. If ρ is integer then there are five alternatives [2, 59] arising from these relations. The purpose of this note is to discuss completely—by means of suitable examples where necessary—whether the functions $f(z)$ belong to one or more types when one of this alternative holds.

2. Suppose first that ρ is an integer.

Case (A): $p \leqslant \rho_1 < \rho = q$.

$f(z)$ belongs to mean type [2, 71-72].

2·1. Case (B): $\rho_1 = \rho = p = q$.

In this case $f(z)$ may belong to maximum or mean type or minimum type, class of divergence. For consider

(i) $f(z)=e^z\,P(z)=e^z\prod_{1}^{\infty} E\left(\frac{z}{-n}, 1\right)$

$$\log M(r, f) \sim \log M(r, P) \sim r \log r$$

and hence $f(z)$ belongs to maximum type, order 1.

(ii) $e^z \sin z$ is of order one and mean type;

(iii) $f(z) = \prod_{2}^{\infty}\left\{1 + \frac{z}{n(\log n)^{3/2}}\right\} - a$ where a is a constant different from 0 and 1. It follows easily that $f(z)$ is of minimum type, order one, and that

$$f(z)=(1-a)\,e^{\alpha z}\prod_{1}^{\infty} E\left(\frac{z}{a_n}, 1\right)$$

where α is a constant different from zero.

We note that† $\sum 1/r_n^{p} = \sum 1/r_n^{\rho}$ is divergent and so $\int^{\infty}\frac{n(r)\,dr}{r^{\rho+1}}$ and $\int^{\infty}\frac{\log M(r)\,dr}{r^{\rho+1}}$ are both divergent [2, 182-184]. Hence every function of minimum type, for which condition (B) is satisfied, must belong to the class of divergence.

2·2. Case (C): $q<\rho=p=\rho_1$.

Here also the function may belong to maximum or mean type or minimum type class of divergence; for $\prod_{1}^{\infty} E\left(\frac{z}{-n}, 1\right)$ is of maximum type, and $\sin z$ of mean type order 1. Further $f(z) = \prod_{2}^{\infty}\left\{1 - \frac{z^2}{n^2\log^2 n}\right\}$ is of minimum type order 1 and class of divergence.

2·3. Case (D): $\rho_1=\rho=q$; $p=\rho-1$.

We shall prove that $f(z)$ must belong to mean type.

LEMMA‡ *Let $P(z)$ be a canonical product of order ρ and genus $p=\rho-1$. If about each zero a_n ($|a_n|>1$) we describe a circle of radius $|a_n|^{-h}$ where $h>\rho$ then in the region excluded from these circles*

$$\log|P(z)| > -\varepsilon r^{\rho} \qquad (r > r_0(\varepsilon))$$

† $r_n = |a_n|$; $n(r)$ is the number of zeros of $f(z)$ in $|z| \leq r$.

‡ The above proof is new. Another proof is given by Pringsheim [1 790-794 (794)]. Our proof is similar to that of a known theorem [2, 57] [3, 273].

Corrigenda to

A NOTE ON THE CLASSIFICATION OF INTEGRAL FUNCTIONS

BY

S. M. SHAH

The lemma on p. 64 lines 21-24 and p. 65, the footnote marked ‡ on p. 64 and the footnote on p. 65 are not necessary, and should be replaced by the following :—

"$f(z)$ cannot belong to maximum type [2, 60]. If $f(z)$ did belong to minimum type, then since $\rho_1 = \rho$; $p = \rho - 1$ it would follow that $q \leqslant \rho - 1$ [2, 90]. This contradicts our hypothesis $q = \rho$. Hence $f(z)$ belongs to mean type."

For, Suppose first $\rho > 1$.

$$\log |P(z)| \geqslant \sum \log \left|1 - \frac{z}{a_n}\right| - \sum O\left\{\left(\frac{r}{r_n}\right)^p\right\}$$

$$- \sum O\left(\frac{r^{p+1}}{r^p_n\,(r + r_n)}\right) = \sum_1 - \sum_2 - \sum_3$$ say; where the summation extend over $r_n \leqslant r\,(\log r)^{-1/\rho}$ for $\sum_1$ and $\sum_2$ and over

$$r_n > r\,(\log r)^{-1/\rho} \text{ for } \sum_3.$$

Now, $\sum_2 = O\,\{r^{p+1}\,(\log r)^{-1/\rho}\} = o\,(r^\rho)$

$$\sum_3 = O\left(r^{p+1}\int_0^\infty \frac{n(t)\,dt}{t^{p+1}\,(t+r)}\right) = o\,(r^\rho)$$

Now if $r_n \leqslant 1$ then for $r > R_0$ we have $\log\left|1 - \frac{z}{a_n}\right| > 0$

If Z lies outside all circles $|z - a_n| = r_n^{-h}$ and $1 < r_n \leqslant r\,(\log r)^{1-/\rho}$ then

$$\left|1 - \frac{z}{a_n}\right| > r_n^{-1-h} > r^{-1-h}\,(\log r)^{(1+h)/\rho} > r^{-1-h}.$$

Hence $\qquad \sum_1 > -(1+h)\,\log r\; n\,(r\,(\log r)^{-1/\rho})$

$$= -(1+h)\log r\,\frac{n(r\,(\log r)^{-1/\rho})}{r^\rho\,(\log r)^{-1}}\,r^\rho\,(\log r)^{-1}$$

$$> -\frac{\varepsilon}{3}\,r^\rho \qquad (r \geqslant R_1\,(\varepsilon))$$

Hence for $r \geqslant r_0\,(\varepsilon) > \text{Max}\,(R_0, R_1)$

$$\log |P(z)| > -\varepsilon r^\rho$$

If $\rho = 1$, then

$$\log |P(z)| = \sum \log \left|1 - \frac{z}{a_n}\right| + \sum \log \left|1 - \frac{z}{a_n}\right|$$
$$= \sum_1 + \sum_2,$$

where the summation extends over $r_n < r/\log r$ for $\sum_1$ and over $r_n \leqslant r/\log r$ for $\sum_2$; and

$$\sum_2 = O\left(r\int_{r/(\log r)^{-1}}^\infty \frac{n\,(t)\,dt}{t\,(t+r)}\right) = o\,(r)$$

Hence the required result follows as before.

From the Lemma we easily obtain that $f(z)$ is of mean type*

* In fact it follows that $\log M(r)/r^\rho$ tends to $|b|$, as $r \to \infty$, if $g(z) = bz^q + O\,(z^{q-1})$.

2·4. Case (E):

$\rho_1=\rho$; $p=\rho-1$, $q<\rho$. Here $\log M(r)/r^{\rho} \to 0$ and hence $f(z)$ is of minimum type. To show that $f(z)$ may belong to either class, we consider

$$f(z, \alpha)=\prod_{2}^{\infty}\left\{1+\frac{z}{n(\log n)^{\alpha}}\right\} \qquad (\alpha>1)$$

for which

$$\log M(r, f) \sim \frac{1}{\alpha-1}\, r(\log r)^{1-\alpha}$$

Hence $f(z, \alpha)$ belongs to the class of convergence if $\alpha>2$ and to the class of divergence if $1<\alpha\leqslant 2$.

3. We suppose now that ρ is not an integer. We have $\rho_1=\rho>0$, $q\leqslant[\rho]=p$. To show that $f(z)$ may belong to any type we consider [4], [5, 217-221].

$$f(z, \alpha)=\prod_{2}^{\infty} E\left(\frac{z}{a_\nu}, p\right);\ a_\nu=-[\nu(\log \nu)^{\alpha}]^{1/\rho};\ p<\rho<p+1;\ \alpha \text{ real}$$

$$\log M(r, f) \sim kr^{\rho}(\log r)^{-\alpha}$$

Hence $f(z, \alpha)$ belongs to maximum type if $\alpha<0$, to mean type if $\alpha=0$, to the minimum type class of divergence if $1\geqslant\alpha>0$ and to the class of convergence if $1<\alpha$.

Let $f_k(z)$ be the kth derivative of $f(z)$; and let $M_k(r)$ and $n_k(r)$ refer to $f_k(z)$. Since, for all large r we have

$$\{M(r)-c\}/r < M_1(r) < M(r+1)$$

it follows that $f(z)$ and $f_1(z)$, and hence all $f_k(z)$, belong to the same order, type and class.

Let $\phi(x)$ be any positive L function such that $\int^{\infty}\frac{dx}{x\,\phi(x)}$ is convergent. If [6] $f(z)$ belongs to non-integral order, or to maximum or minimum type integral order, then

(A) $$\varliminf_{r=\infty} \frac{\log M(r)}{n(r)\,\phi(r)}=0$$

and hence under the same conditions on $f(z)$, we have

(B) $$\varliminf_{r=\infty} \frac{\log M_k(r)}{n_k(r)\,\phi(r)}=0 \qquad (k=1, 2, 3 \ldots\ldots)$$

If $f(z)$ belongs to mean type integral order, (A) and (B) may not hold. For instance let $\phi(x)=\log^2 x$.

Ex. I. $f(z)=ze^{z}$; (A) and (B) are false

" II. $f(z)=\sin z$; (A) and (B) hold

" III. $f(z)=e^{z}-1$; (A) holds but (B) is false.

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GLEANINGS

"$h\nu$"

All black body radiations
All the spectrum variations,
All atomic oscillations
Vary as "$h\nu$"

CHORUS

Here's the right relation
Governs radiation,
Here's the new
And only true
Electrodynamical equation:
Never mind your d^2/dt^2
Ve or half mv^2
(If you watch the factor "c^2",
's equal to "$h\nu$".

Ultraviolet vibrations,
X and gamma ray pulsations
Ordinary light sensations
All obey "$h\nu$":

Even in matters calorific,
Such things as the heat specific
Yield to treatment scientific
If you use "$h\nu$".

In all questions energetic,
Whether static or kinetic,
Or electric or magnetic,
You must use "$h\nu$".

There would be a mighty clearance,
We should all be Planck's adherents,
Were it not that interference
Still defies "$h\nu$"

G. S.

From G. F. HULL'S *Elementary Survey of Modern Physics.*

A PROOF OF HADAMARD'S FACTORIZATION THEOREM

BY

C. T. RAJAGOPAL,* *Madras Christian College, Tambaram.*

1. There are two simple proofs of Hadamard's theorem—(*a*) the direct proof of Landau reproduced by Titchmarsh [(3), §8·24]†, (*b*) the indirect proof which employs the minimum-modulus theorem for a canonical product (Titchmarsh, *loc. cit.*, §8·72). This article aims at bringing to the fore a point somewhat obscured by the two proofs: Hadamard's theorem issues directly from considerations based on the structure of a canonical product, considerations which lead at the same time to the other inequalities between the real order, the apparent order and the genus of an integral function. These considerations are embodied in the theorem in §2. Although the theorem is not new in content‡, its proof is given in full to make the argument run continuously.

1·1. The lemmas required in our proofs are collected below. In the first two, $f(z)$ represents an integral function with $f(0)\neq 0$ and an infinity of zeros arranged in a sequence $z_1, z_2, z_3, \ldots$ so that

$$0 < |z_1| \leqslant |z_2| \leqslant \cdots \leqslant |z_n| \to \infty \ (n \to \infty).$$

LEMMA 1.—*The real order* (ρ_1) *of* $f(z)$, *or the convergence exponent of* $\{|z_n|\}$, *cannot exceed the (apparent) order* (ρ) *of* $f(z)$; *i.e.*, $\rho_1 \leqslant \rho$. [HADAMARD.]

LEMMA 2.—*If* ρ (*and hence* ρ_1) *is finite, let* p *be the largest integral value of* τ *for which* $\sum |z_n|^{-\tau}$ *is divergent; then*

$$f(z) = e^{H(z)}\, G(z),$$

where $H(z)$ *is an integral function and*

$$G(z) = \prod_{n=1}^{\infty} E\left(\frac{z}{z_n}, p\right);$$

$E(u, p)$ *denoting* $(1-u)\, e^{u+u^2/2+\cdots+u^p/p}$ *or* $(1-u)$ *according as* $p>0$ *or* $=0$. [WEIERSTRASS]

* I wish to thank Dr. V. Ganapathy Iyer who, by his suggestions, has clarified some of my original statements.

† Another version of the proof is given by Ingham [(2), Chap. iii, §7, Theorem F3].

‡ *Cp.* for instance Copson [(1), § 7·51].

If $H(z)$ is a polynomial, let q be its degree; then the genus $(\bar{p})$ of $f(z)$, i.e., max (p, q), is finite.

LEMMA 3.—*Any integral function $H(z)$ for which $\mathbf{R}H(z)=O(|z|^k)$ on a sequence of circles of arbitrarily large radius, must be a polynomial of degree not greater than k.* [HADAMARD.]

LEMMA 4. *If $0<|u|<1/2$, $p\geqslant 0$, $\tau\leqslant p+1$, then*

$$|E(u, p)| \begin{cases} < \exp(2|u|^{\tau}), & \quad \dots (1) \\ > \exp(-2|u|^{\tau}). & \quad \dots (2) \end{cases}$$

Proof.

$$\left| E(u, p) \right| = \left| \exp\left(-\frac{u^{p+1}}{p+1} - \frac{u^{p+2}}{p+2} - \cdots \right) \right|$$
$$= \exp\left\{ \mathbf{R}\left(-\frac{u^{p+1}}{p+1} - \frac{u^{p+2}}{p+2} - \cdots \right) \right\}.$$

The proof is completed by observing that

$$\left| \mathbf{R}\left(-\frac{u^{p+1}}{p+1} - \frac{u^{p+2}}{p+2} - \cdots \right) \right| \leqslant \left| \frac{u^{p+1}}{p+1} + \frac{u^{p+2}}{p+2} + \cdots \right|$$
$$< |u|^{p+1}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots \right)$$
$$\leqslant 2|u|^{\tau}.$$

LEMMA 5.—*If $|u|\geqslant 1/2$, $0\leqslant p<\tau$, then*

$$|E(u, p)| < \exp(A|u|^{\tau}), \quad \dots (3)$$

$$\left| \exp\left(u + \frac{u^2}{2} + \cdots \frac{u^p}{p} \right) \right| > \exp(-A|u|^{\tau}). \quad \dots (4)$$

where $A>0$ is a number depending only on τ and on p.

Proof.

$$|E(u, p)| < (1+|u|)\exp\left(|u| + \frac{|u|^2}{2} + \cdots \frac{|u|^p}{p} \right)$$
$$= \exp\left\{ \log(1+|u|) + |u| + \cdots \frac{|u|^p}{p} \right\}$$
$$= \exp\{F(|u|)\,|u|^{\tau}\},$$

where $F(|u|)$ is a positive continuous function for $|u|\geqslant 1/2$, tending to 0 as $|u|\to\infty$ and hence $\leqslant A$. This leads to (3).

(4) follows from:

$$\mathbf{R}\left(u + \frac{u^2}{2} + \cdots \frac{u^p}{p} \right) \geqslant -|u| - \frac{|u|^2}{2} - \cdots \frac{|u|^p}{p}$$
$$> -F(|u|)\,|u|^{\tau}.$$

2. THEOREM. *Let τ be such that $0 \leqslant p < \tau \leqslant p+1$ and $\sum |z_n|^{-\tau}$ is convergent. Also let N be a positive integer such that $|z_N| < |z_{N+1}|$. Then, for z satisfying $|z_N| \leqslant 2|z| < |z_{N+1}|$,*

$$|G(z)| \begin{cases} < \exp\left(B \sum_{n=1}^{\infty} \left|\frac{z}{z_n}\right|^{\tau}\right), & \dots (5) \\ \geqslant \prod_{n=1}^{N} \left|1 - \frac{z}{z_n}\right| \cdot \exp\left(-B \sum_{n=1}^{\infty} \left|\frac{z}{z_n}\right|^{\tau}\right), & \dots (6) \end{cases}$$

where $B > 0$ is a number depending only on τ and on p.

Proof.

$$|G(z)| = \prod_{n=1}^{N} \left| E\left(\frac{z}{z_n}, p\right) \right| \cdot \prod_{n=N+1}^{\infty} \left| E\left(\frac{z}{z_n}, p\right) \right|.$$

Employing (3) in $\prod_{n=N+1}^{\infty} | \cdots |$ and (1) in $\prod_{n=1}^{N} | \cdots |$, we get (5), B denoting max $(A, 2)$.

Again,

$$|G(z)| = \prod_{n=1}^{N} \left|1 - \frac{z}{z_n}\right| \cdot \prod_{n=1}^{N} \left| \exp\left(\frac{z}{z_n} + \cdots \frac{z^p}{p z_n^p}\right) \right| \prod_{n=N+1}^{\infty} \left| E\left(\frac{z}{z_n}, p\right) \right|$$

Using (4) in $\prod_{n=1}^{N} |\exp(\cdots)|$ and (2) in $\prod_{n=N+1}^{\infty} |\cdots|$, we obtain (6)*.

In (5) and (6) we can take $\tau = \rho_1 + \varepsilon < p+1$ or $\tau = \rho_1$ according as $\rho_1 \neq p+1$ or $= p+1$. We have then the

COROLLARY.

$$|G(z)| \begin{cases} < \exp\left(C |z|^{\rho_1 + \varepsilon}\right), & \dots (7) \\ \geqslant \prod_{n=1}^{N} \left|1 - \frac{z}{z_n}\right| \cdot \exp\left(-C |z|^{\rho_1 + \varepsilon}\right). & \dots (8) \end{cases}$$

where $C = B \sum_{n=1}^{\infty} |z_n|^{-\rho_1 - \varepsilon}$.

2·1. We can deduce from our theorem the following inequalities (in every one of which we presuppose the existence of the larger member).

(i) $\rho \leqslant \max(\rho_1, q)$ [BOREL],

which implies $\rho \leqslant \bar{p} + 1$;

* In (6), the equality sign occurs when z is a zero of $f(z)$.

(ii) $|G(z)| < \exp(\varepsilon |z|^{p+1})$ *for* $|z| \geqslant R_0(\varepsilon)$ [POINCARÉ];

(iii) $\rho \geqslant \max(\rho_2, q)$ [HADAMARD],

which, in conjunction with (i), makes $\rho = \max(\rho_2, q)$;

(iv) $|G(z)|^{-1} < \exp(|z|^{\rho_1+\varepsilon})$ *on circles of arbitrarily large radius.* [BOREL].

The reader may refer to Ingham (*loc. cit.*, Theorem F 1), Vivanti [(4), § 183], Copson (*loc. cit.*, pp. 173-4) for the proofs of (i), (ii), (iv) respectively. (iii) is merely a combination of Lemma 1 and Hadamard's factorization theorem, $q \leqslant \rho$.

3. *Proof of* $q \leqslant \rho$. We can find a sequence of circles, centre the origin, not passing through the zeros of $f(z)$, with radii:

$$R_{n_1} < R_{n_2} < \cdots < R_{n_\nu} \to \infty \ (\nu \to \infty)$$

$\{n_\nu\}$ being an increasing sequence of positive integers such that $|z_{n_\nu}| \leqslant 2 R_{n_\nu} < |z_{n_\nu+1}|$.

On the circle $|z| = R_N$ of this sequence

$$\begin{aligned}|f(z)| &= \exp\{\mathbf{R}\, H(z)\} \cdot |G(z)| \\ &> \exp\left\{\mathbf{R}\, H(z) - C|z|^{\rho_1+\varepsilon}\right\} \cdot \prod_{n=1}^{N} \left|1 - \frac{z}{z_n}\right|, \qquad \ldots (9)\end{aligned}$$

using successively Lemma 2 and (8).*

On $|z| = 4R_N$,

$$\frac{|f(z)|}{\prod_{n=1}^{N}\left|1 - \frac{z}{z_n}\right|} \leqslant \frac{M(4R_N)}{\prod_{n=1}^{N}\left\{(4R_N/2R_N) - 1\right\}}$$

$$= \exp\left\{O(4R_N)^{\rho+\varepsilon}\right\} \ (R_N \to \infty) \quad \ldots (10)$$

By the maximum-modulus principle (10) holds for $|z| = R_N$ and can be employed in (9) so as to yield

$$\exp.\left\{\mathbf{R}\, H(z) - C|z|^{\rho_1+\varepsilon}\right\} = \exp\left\{O(|z|^{\rho+\varepsilon})\right\} (|z| = R_N \to \infty).$$

Lemma 3, in conjunction with Lemma 1, now shows that $H(z)$ is a polynomial of degree $\leqslant \rho + \varepsilon$ whence, ε being arbitrarily small, Hadamard's theorem follows.

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* Since $|z| = R_N$ does not pass through a zero of $f(z)$ we note that $\prod_{n=1}^{N} |\cdots| \neq 0$ and omit the equality sign in (8).

REMARK. It may be observed that $G(z)$ in the proof of (iv) is subjected to an analysis more detailed than the one required to establish that $q \leqslant \rho$. Hence the proof (*b*) of Hadamard's result $q \leqslant \rho$. by means of Borel's (iv), really involves an argument in the nature of a *detour*. The proof suggested here is an attempt to avoid the *detour*.

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GLEANINGS

Poetry, in fact, bears the same kind of relation to Prose, that Algebra bears to Arithmetic......... The test of a poet is the frequency and diversity of the occasions on which we remember his poetry,

From AUDEN & GARRETT:
The Poets Tongue, p. vi.

CARATHÉODORY'S INEQUALITY AND ALLIED RESULTS

BY

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1. Theorem 1 of this paper is Carathéodory's inequality in one of its forms. Theorems 2, 3, 4 are known results allied to Theorem 1. The usual proofs of the theorems involve directly or indirectly the maximum-modulus principle. Titchmarsh [(5); §§ 5·5, 5·51], following Landau [(3); §73], establishes inequalities deducible from Theorems 1, 2 by an appeal to Schwarz's lemma which is an immediate consequence of the maximum-modulus principle. Pólya-Szegö [(4): III Abschn., Nr. 283-285] adopt much the same point of view with regard to Theorems 1, 3, 4. Ingham [(2): Chap. III, Theorem E] proves Theorems 1, 2 in forms which do not require the use of the maximum-modulus principle; but the equivalence of these forms and the forms given here is not obvious without the principle. In view of the fact that the maximum-modulus principle can itself be regarded as a consequence of Cauchy's fundamental (or integral) formula [(1): § 7·3; (5): § 5·11] it would seem pertinent to ask whether Theorems 1-4 cannot be proved by an appeal to the fundamental formula and nothing beyond. What follows is an attempt to show that such a question admits of an affirmative answer.

NOTATION. $f(z)$ is a function analytic in a region including $|z| \leqslant R$. $w = Re^{\phi i}$, $z = re^{\theta i}$ are points respectively on and inside $|z| = R$. $\mathbf{R}f(Re^{\phi i}) = P(R, \phi)$; $\max\limits_{|z|=r} |f(z)| = M(r)$, $\max\limits_{|z|=r} \mathbf{R}f(z) = A(r)$.

2. The following lemmas will be employed in our proofs.

LEMMA 1. $$\mathbf{R}f(0) = \frac{1}{2\pi}\int_0^{2\pi} P(R, \phi)\, d\phi.$$

This follows at once from

$$f(0) = \frac{1}{2\pi i}\int_{|w|=R} \frac{f(w)}{w}\, dw = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{\phi i})\, d\phi.$$

From the lemma we can deduce the general Jensen formula.

Let $\phi(z)$ be meromorphic in $|z| \leqslant R$ with zeros a_λ $(\lambda = 1, 2, \ldots\ldots l)$ and poles $b_\mu (\mu = 1, 2, \ldots\ldots m)$. Then

$$\log|\phi(0)| + \log\left\{ R^{l-m} \left| \frac{b_1 b_2 \cdots b_m}{a_1 a_2 \cdots a_l} \right| \right\} = \frac{1}{2\pi}\int_0^{2\pi} \log|\phi(Re^{\phi i})|\, d\phi,$$

where a zero or pole of order p is counted p times.

This is proved by taking $f(z) = \log F(z)$, where

(1) $$F(z) = \phi(z) \cdot \prod_{\lambda=1}^{l} \frac{R^2 - \bar{a}_\lambda z}{(z - a_\lambda)R} \cdot \prod_{\mu=1}^{m} \frac{(z - b_\mu)R}{(R^2 - \bar{b}_\mu z)},$$

the bar above a complex number denoting the conjugate of the complex number.

LEMMA 2. $$\int_{|w|=R} \frac{\overline{f(w)}}{w^{n+1}}\, dw = 0 \qquad (n = 1, 2, 3, \ldots\ldots)$$

Proof. Since $\int_{|w|=R} f(w) w^n\, dw = 0$ $(n = 0, 1, 2, \ldots\ldots)$, the conjugate of the integral is zero; i.e.,

$$\int_{|w|=R} \overline{f(w)}\ \overline{w^n}\ \overline{dw} = 0 \qquad (n = 0, 1, 2, \ldots\ldots)$$

This is equivalent to the lemma.

LEMMA 3. $$f(z) - f(0) = \frac{1}{\pi i}\int_{|w|=R} \frac{z}{w(w-z)}\, \mathbf{R} f(w)\, dw$$

(2) $$= \frac{1}{\pi}\int_0^{2\pi} \frac{re^{\theta i}}{Re^{\phi i} - re^{\theta i}} P(R, \phi)\, d\phi.$$

Proof.

$$\frac{1}{2\pi i}\int_{|w|=R} \frac{z}{w(w-z)} f(w)\, dw = \frac{1}{2\pi i}\int_{|w|=R} \left(\frac{1}{w-z} - \frac{1}{w}\right) f(w)\, dw$$

(3) $$= f(z) - f(0).$$

Also,

$$\frac{1}{2\pi i}\int_{|w|=R} \frac{z}{w(w-z)} \overline{f(w)}\, dw = \frac{1}{2\pi i}\int_{|w|=R} \overline{f(w)} \sum_{n=1}^{\infty} \frac{z^n}{w^{n+1}}\, dw$$

$$= \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{|w|=R} \overline{f(w)} \frac{z^n}{w^{n+1}}\, dw$$

(where the term-wise integration is justified by uniform convergence)

(4) $$= 0,$$

in virtue of the Lemma 2.

Adding (3) and (4), we obtain Lemma 3.

Equating real parts of both sides of (2) and employing Lemma 1, we are led to Poisson's integral formula [(5) : § 3·6].

Again Poisson's formula applied to $\log F(z)$, where $F(z)$ is as in (1), gives the *Poisson-Jensen formula* [(5) : § 3·62].

LEMMA 4. $$\frac{f^{(\nu)}(z)}{\nu!}=\frac{1}{\pi i}\int_{|w|=R}\frac{\mathbf{R}f(w)}{(w-z)^{\nu+1}}\,dw \qquad (\nu=1, 2, \ldots)$$

(5) $$=\frac{1}{\pi}\int_0^{2\pi}\frac{Re^{\phi i}}{(Re^{\phi i}-re^{\theta i})^{\nu+1}}P(R,\phi)\,d\phi.$$

Proof.

(6) $$\frac{1}{2\pi i}\int_{|w|=R}\frac{f(w)}{(w-z)^{\nu+1}}\,dw=\frac{f^{(\nu)}(z)}{\nu!}.$$

$$\frac{1}{2\pi i}\int_{|w|=R}\frac{\overline{f(w)}}{(w-z)^{\nu+1}}\,dw=\frac{1}{2\pi i}\int_{|w|=R}\overline{f(w)}\sum_{n=1}^{\infty}\frac{a_n z^n}{w^{n+\nu+1}}\,dw$$
$$=\frac{1}{2\pi i}\sum_{n=0}^{\infty}a_n z^n\int_{|w|=R}\frac{\overline{f(w)}}{w^{n+\nu+1}}\,dw$$

(7) $$=0.$$

Adding (6) and (7) we get the lemma.

Equating real parts of both sides of (5), we find

$$\mathbf{R}\,\frac{f^{(\nu)}(z)}{\nu!}=\frac{1}{\pi}\int_0^{2\pi}\frac{R\sum_{s=0}^{\nu+1}(-1)^s\binom{\nu+1}{s}R^{\nu+1-s}r^s\cos(\overline{\nu-s}\,\phi+s\theta)}{\{R^2-2Rr\cos(\theta-\phi)+r^2\}^{\nu+1}}\times P(R,\phi)\,d\phi.$$

3. THEOREM 1.

$$|f(z)|\leqslant|f(0)|+\frac{2r}{R-r}\{A(R)-\mathbf{R}f(0)\}.$$

Proof. From (2), changing $f(z)$ to $A(R)-f(z)$, we get

(8) $$-f(z)=-f(0)+\frac{1}{\pi}\int_0^{2\pi}\frac{re^{\theta i}}{Re^{\phi i}-re^{\theta i}}\{A(R)-P(R,\phi)\}\,d\phi$$

whence

$$|f(z)|\leqslant|f(0)|+\frac{1}{\pi}\int_0^{2\pi}\frac{r}{R-r}\{A(R)-P(R,\phi)\}\,d\phi.$$

This, in conjunction with Lemma 1, gives the theorem.

COROLLARY.

$$|f(z)|\leqslant\frac{R+r}{R-r}|f(0)|+\frac{2r}{R-r}A(R);\text{[1]}$$

or,

$$M(r)\leqslant\frac{R+r}{R-r}M(0)+\frac{2r}{R-r}A(R).$$

THEOREM 2.

$$\left|\frac{f^{(\nu)}(z)}{\nu!}\right|\leqslant\frac{2R}{(R-r)^{\nu+1}}\{A(R)-\mathbf{R}f(0)\} \qquad (\nu\geqslant1).$$

---

[1] Titchmarsh states the result in the form:

$$|f(z)|\leqslant\frac{R+r}{R-r}\left\{|f(0)|+A(R)\right\}$$

which is true only for $A(R)\geqslant 0$. For instance, when $f(z)=-K<0$, it is obviously false.

Proof. We start from (5) and proceed exactly on the lines of the proof of Theorem 1.

Putting $z=0$ in the theorem, we deduce the

COROLLARY. If $f(z) = \sum_{n=0}^{\infty} c_n z^n \ (0 \leqslant |z| \leqslant R)$, then

(9) $$| c_\nu | \leqslant 2\{A(R) - \mathbf{R} f(0)\}/R^\nu \qquad (\nu \geqslant 1)^1.$$

THEOREM 3.

$$\mathbf{R} f(z) \leqslant \mathbf{R} f(0) + \frac{2r}{R+r} \{A(R) - \mathbf{R} f(0)\}.$$

Proof. Equating real parts of both sides of (8), we obtain

$$-\mathbf{R}\ f(z) = -\mathbf{R}\ f(0) + \frac{1}{\pi}\int_0^{2\pi} \frac{Rr\cos(\theta-\phi) - r^2}{R^2 - 2Rr\cos(\theta-\phi) + r}\ \{A(R) - P(R,\phi)\}d\phi$$

$$\geqslant -\mathbf{R} f(0) - \frac{1}{\pi}\int_0^{2\pi} \frac{r}{R+r}\{A(R) - P(R,\ \phi)\}\ d\phi$$

which is equivalent to the result stated.

COROLLARY.

$$A(r) \leqslant \frac{R-r}{R+r}\ A(0) + \frac{2r}{R+r} A(R).$$

THEOREM 4.

$$|f(z)| \leqslant |f(0)|^{(R-r)/(R+r)} \{M(R)\}^{2r/(R+r)},$$

provided $f(z)$ has no zeros in $|z| \leqslant R$.

Proof. The theorem results on the application of the previous one to $\log f(z)$.

COROLLARY 1. *If $f(z)$ has zeros $a_\lambda \neq 0$ $(\lambda = 1,\ 2, \ldots l)$ in $|z| < R$, a zero of order p being counted p times, then*

(10) $$|f(z)| \prod_{\lambda=1}^{l} \left| \frac{R^2 - \bar{a}_\lambda z}{(z - a_\lambda)R} \right| \leqslant \left\{ \frac{|f(0)|R^l}{|a_1 a_2 \cdots a_l|} \right\}^{(R-r)/(R+r)} \times$$

$$\{M(R)\}^{2r/(R+r)} \quad (0 \leqslant |z| < R),$$

where, if $f(z)$ has a simple zero at $z = a_\lambda$, we define $\{f(z)/(z - a_\lambda)\}_{z=a_\lambda}$ as $f'(a_\lambda)$.

This is proved by taking instead of $f(z)$ in Theorem 4 the function :

$$f(z) \cdot \prod_{\lambda=1}^{l} \frac{R^2 - \bar{a}_\lambda z}{(z - a_\lambda)R}.$$

(10) gives Jensen's inequality when $z=0$. We can make it include Schwarz's lemma also by stating it in the form :

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[1] This result is stated in a somewhat different form by Titchmarsh [(5): § 2·53] Theorems 1 and 2 can be derived from (9) by assuming that $f(z)$ is expansible in a power series in $|z| < R$ and defining $A(r)$, $A(R)$ respectively as $\max_{|z| \leqslant r} \mathbf{R} f(z)$, $\lim_{r \to R-0} A(r)$ [(2): Theorem E]

COROLLARY 2. *If $f(z)$ has a simple zero at $z=0$ in addition to the zeros in Corollary* 1, *then*

$$(11)\qquad \left|\frac{f(z)}{z}\right| R\cdot\prod_{\lambda=1}^{l}\left|\frac{R^2-\bar{a}_\lambda z}{(z-a_\lambda)R}\right| \leqslant \left\{\frac{|f'(0)|R^{l+1}}{|a_1 a_2\cdots a_l|}\right\}^{(R-r)/(R+r)}\times \{M(R)\}^{2r/(R+r)} \quad (0\leqslant |z|<R).$$

(10) and (11) are sharper forms of (12) below [*Cp.* (4): III Abschn., Nr. 294] which contains Schwarz's lemma and Jensen's inequality as particular cases.

$$(12)\qquad |f(z)|\cdot\prod_{\lambda=1}^{l}\left|\frac{R^2-\bar{a}_\lambda z}{(z-a_\lambda)R}\right| \leqslant M(R) \qquad (0\leqslant |z|<R).$$

It is obvious that the inequalities (10), (11), (12) can be extended to the meromorphic function $\phi(z)$ in (1) by applying Theorem 4 to the function $F(z)$.

The theorems discussed above achieve in different ways their common objective of connecting a value of $f(z)$ inside a region of analyticity of $f(z)$ with a value or values of $f(z)$ on the boundary of the region. It emerges from the discussion that all such theorems may be viewed as direct consequences of Cauchy's fundamental formula[1].

References.

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[1] Cauchy's fundamental formula without the Cauchy-Taylor theorem leads to the maximum-modulus principle in the form:

If $f(z)$ is analytic on and inside a simple closed contour C and $|f(z)|\leqslant M$ on C, then $|f(z)|\leqslant M$ inside C. [*Vide* Landau's proof of the principle reproduced in Copson, *loc. cit.*, or in Pólya-Szegö *loc. cit.*, IV Abschn., P 31.]

A NOTE ON THE THEORY OF INFINITE SERIES

BY

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The present note arises out of the consideration of a paper by Mr. C. T. Rajagopal in *The Mathematics Student* [Vol. VIII. No. 3. pp. 118-23]. The problem under discussion is:

Given an infinite series $\sum a_n$ *and an increasing divergent sequence of positive numbers* D_n

$$0 = D_0 < D_1 < D_2 < \ldots\ldots < D_n \to \infty$$

$$d_n = D_n - D_{n-1}$$

under what circumstances can it be said that the condition

$$\lim_{n \to \infty} \frac{a_n D_n}{d_n} = 0 \qquad \ldots \quad (1)$$

is necessary for the convergence of $\sum a_n$*?*

If $d_n \equiv 1$ the problem becomes: *Under what circumstances is* $\lim na_n = 0$ *a necessary condition for the convergence of* $\sum a_n$*?*

Our answer to the above problem is to be sought from the following observations: If

$$b_n = D_n \left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} \right)$$

then $$b_n = \frac{a_n D_n}{d_n} - \frac{a_{n+1} D_{n+1}}{d_{n+1}} + a_{n+1} \ldots$$

so that

$$\sum_{r=1}^{n+1} a_r - \sum_{r=1}^{n} b_r = \frac{a_{n+1} D_{n+1}}{d_{n+1}} \qquad \ldots \quad (2)$$

A series $\sum a_n$ with partial sums $A_n \left(= \sum_{r=1}^{n} a_r \right)$ is said to be summable (D 1) if

$$\lim_{n=\infty} \frac{\sum_{r=1}^{n} d_{r+1} A_r}{D_{n+1}}$$

exists. If a series $\sum a_n$ is convergent, it is also summable (D 1).

Now if $\sum a_n$ is convergent, then

$$\lim \frac{\sum_{1}^{n} a_r D_r}{D_n} = 0 \qquad \ldots (3)$$

and if we set $x_n = \frac{a_n D_n}{d_n}$, then (3) implies

$$\lim \frac{\sum_{r=1}^{n} d_r x_r}{D_n} = 0 \qquad \ldots (4)$$

Therefore, from (2) and (4), we see, that if $\sum a_n$ is convergent, the series whose partial sums are $\sum_{1}^{n+1} a_r - \sum_{1}^{n} b_r$ is summable (D 1) to the sum 0. Hence $\sum b_r$ is summable (D 1) to the sum of the series $\sum a_n$. Hence we have

THEOREM 1. *If* $\sum a_n$ *is convergent, then the series*

$$\sum D_n \left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}}\right)$$

is summable (D 1).

On the other hand from (2) we observe the following

THEOREM 2. *If* $\sum D_n \left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}}\right)$ *is convergent, then* $\sum a_n$ *will be convergent if and only if,* $\lim \frac{a_n D_n}{d_n} = 0$.

THEOREM 3. *If* $\sum a_n$ *is convergent then* $\lim \frac{a_n D_n}{d_n} = 0$ *if and only if* $\sum D_n \left(\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}}\right)$ *is convergent.*

Theorems 1 and 3 show that any Tauberian condition satisfied by $\sum b_n$ will ensure its convergence and then our problem will be solved. Thus, using theorems 1 and 3 and the following lemma:

LEMMA 1. *If* $\sum a_n$ *is summable* (D 1) *it will be convergent if and only if* $\lim \frac{\sum_{r=1}^{n} a_r D_r}{D_n} = 0$

we have the following

THEOREM 4. *If* $\sum a_n$ *is convergent then* $\lim \frac{a_n D_n}{d_n} = 0$ *if and only if*

$$\lim \frac{1}{D_n} \sum_{r=1}^{n} D^2_r \left(\frac{a_r}{d_r} - \frac{a_{r+1}}{d_{r+1}}\right) = 0.$$

We have also the following

THEOREM 5. *If $\sum a_n$ is convergent and if for some $\gamma > 0$*

$$\lim \frac{1}{D_n^{\gamma}} \sum_{n=1}^{n} D_r^{1+\gamma}\left(\frac{a_r}{d_r} - \frac{a_{r+1}}{d_{r+1}}\right) = 0;$$

then $\mathit{lim}\ \frac{a_n D_n}{d_n} = 0.$

This theorem will follow from theorem 4 and the following

LEMMA 2. *If for some $\gamma > 0$*

$$\lim \frac{1}{D_n^{\gamma}} \sum_{r=1}^{n} a_r D_r^{\gamma} = 0,$$

then it will be true for every $\gamma > 0$.

We shall indicate the proof of the integral analogue of this lemma.

LEMMA 3. *If $s(t)$ is of bounded variation in every finite interval contained in $(0, \infty)$ and vanishes identically in a small interval $(0, \delta)$ and if for some $\alpha > 0$*

$$\lim_{\omega=\infty} R_\alpha(\omega) = \frac{1}{\omega^\alpha}\int_0^\omega t^\alpha\, d\,s(t) = 0$$

then for every $\beta > 0$ $\mathit{lim}\ R_\beta(\omega) = 0$.

For

$$R_\beta(\omega) = \frac{1}{\omega^\beta}\int_0^\omega t^\beta\, d\,s(t) = \frac{1}{\omega^\beta}\int_0^\omega t^{\beta-\alpha}\, t^\alpha\, d\,s(t)$$

$$= \frac{1}{\omega^\beta}\left[t^{\beta-\alpha}\int^t u^\alpha ds(u)\right]_0^\omega - \frac{\beta-\alpha}{\omega^\beta}\int_0^\omega t^{\beta-1} R_\alpha(t)\,dt$$

$$= R_\alpha(\omega) - \frac{\beta-\alpha}{\omega^\beta}\int_0^\omega t^{\beta-1} R_\alpha(t)\,dt$$

and the lemma is immediately proved.

REMARKS :—If $a_n \geqslant 0$ and $\frac{a_n}{d_n} \downarrow 0$ then $b_n \geqslant 0$, so that, if $\sum a_n$ is convergent, $\sum b_n$ being summable (D 1) is also convergent. So by theorem 3, $\lim \frac{a_n D_n}{d_n} = 0.$

In addition to the following condition due to C. T. Rajagopal *loc. cit.*

$$\frac{a_n}{d_n} - \frac{a_{n+1}}{d_{n+1}} < O\left(\frac{d_n}{D_n^2}\right)$$

we might mention other Tauberian conditions:

$$\sum_{r=1}^{n} |b_r|^p d_r^{1-p} D_r^p = \sum_{r=1}^{n} d_r^{1-p} D_r^{2p} \left| \frac{a_r}{d_r} - \frac{a_{r+1}}{d_{r+1}} \right|^p$$

$$= O(D_n), \qquad p > 1.$$

$$\sum_{r=1}^{\infty} | b_r |^{\rho+1} D_r^{\rho} d_r^{-\rho} = \sum_{1}^{\infty} d_r^{-\rho} D_r^{2\rho+1} \left| \frac{a_r}{d_r} - \frac{a_{r+1}}{d_{r+1}} \right|^{\rho+1} < \infty, \ \rho > 0.$$

If $d_n = o(D_n)$ one might replace $|b_r|$ by $(|b_r| - b_r)$ in the above conditions. Except for their theoretical interest these latter conditions do not seem to be of practical importance.

Finally, if we put $d_n = 1$ for every n, we have the following corollaries from the above theorems:

COR. 1. *If* $\sum a_n$ *is convergent then* $\sum n(a_n - a_{n+1})$ *is summable* $(C1)$.

COR. 2. *If* $\sum n(a_n - a_{n+1})$ *is convergent then* $\sum a_n$ *will be convergent if and only if* $\lim \cdot na_n = 0$.

COR. 3. *If* $\sum a_n$ *is convergent then* $\lim na_n = 0$ *if and only if* $\sum n(a_n - a_{n+1})$ *is convergent.*

COR. 4. *If* $\sum a_n$ *is convergent then* $\lim na_n = 0$ *if and only if*

$$\lim \frac{1}{n} \sum_{r=1}^{n} r^2(a_r - a_{r+1}) = 0$$

COR. 5. *If* $\sum a_n$ *is convergent and if for some* $\gamma > 0$

$$\lim \frac{1}{n^{\gamma}} \sum_{r=1}^{n} r^{1+\gamma}(a_r - a_{r+1}) = 0$$

then $\lim na_n = 0$.

COR. 6. *If* $\sum a_n$ *is convergent then* $\lim na_n = 0$ *if one of the following conditions are satisfied.*

(i) $n^2 (a_n - a_{n+1}) < O(1)$

(ii) $\sum_{r=1}^{n} r^{2p} |a_r - a_{r+1}|^p = O(n), \ p > 1$

(iii) $\sum_{n=1}^{\infty} n^{2\rho+1} |a_n - a_{n+1}|^{\rho} < \infty, \ \rho > 0.$

OUR FIRST MAGNITUDE STARS VIEWED FROM THE GALACTIC CENTRE

BY

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From time immemorial the stars have been woven into constellations and their differences in brightness noted. Owing to their vast distances these features are fairly permanent being little affected by the motions of our Earth, the Sun and of the stars themselves. Consequently star gazers may fail to realise how changed would be their aspect, were we removed to a considerable distance from the stellar standpoint. It may therefore be of interest to study the effect of our removal to a view point so remote as the galactic centre, which is 10,000 radials (parsecs) away in the direction of Right Ascension 17 hrs. 40 m., Declination $-30°$ on the Ophiuchus Scorpio borderline. A "radial" represents the distance yielding a parallax of 1″ arc and is equal to 3·26 light years, or 19·16 billion (10^{12}) miles.

We shall examine the effect of such a shift of our position on the most familiar stars, the twenty brightest stars of first magnitude. To them we add our own star, the Sun, making twentyone entries in our table. Their proper names appear in the first column, followed by the Greek letter and constellation in the second column. The next two columns give the positions in galactic coordinates according to the plane having its north polar axis towards (1900) R. A. 12 hrs. 48 m. (192°) and Declination $+27°$, this being the mean for all objects defining our Universe; the dynamic centre of this (our view point here) lies at $\lambda = 326°$ $\beta = 0°$ where λ (col. 3) is the galongitude (Galactic longitude) measured eastward from the intersection of our celestial (equatorial) plane with that of the galaxy, at R. A. 18 hrs. 48 m. (282)°, declination 0° in Aquila and β (Col. 4) is the galatitude (galactic latitude) north and south from the galactic equator. The fifth column gives the revised Harvard Photometric magnitude (m) in which order of apparent brightness the stars are tabulated. The next column gives the absolute magnitude (M), or what the stars' magnitude would appear, if removed to the International Astronomical Union's standard distance of 10 radials (parallax 0″·1). A range of 1·0 "magnitude" indicates a 2·5 fold change in brightness; whence it follows that the relation between m and M is $M = m + 5 - 5 \log R$, R being the star's

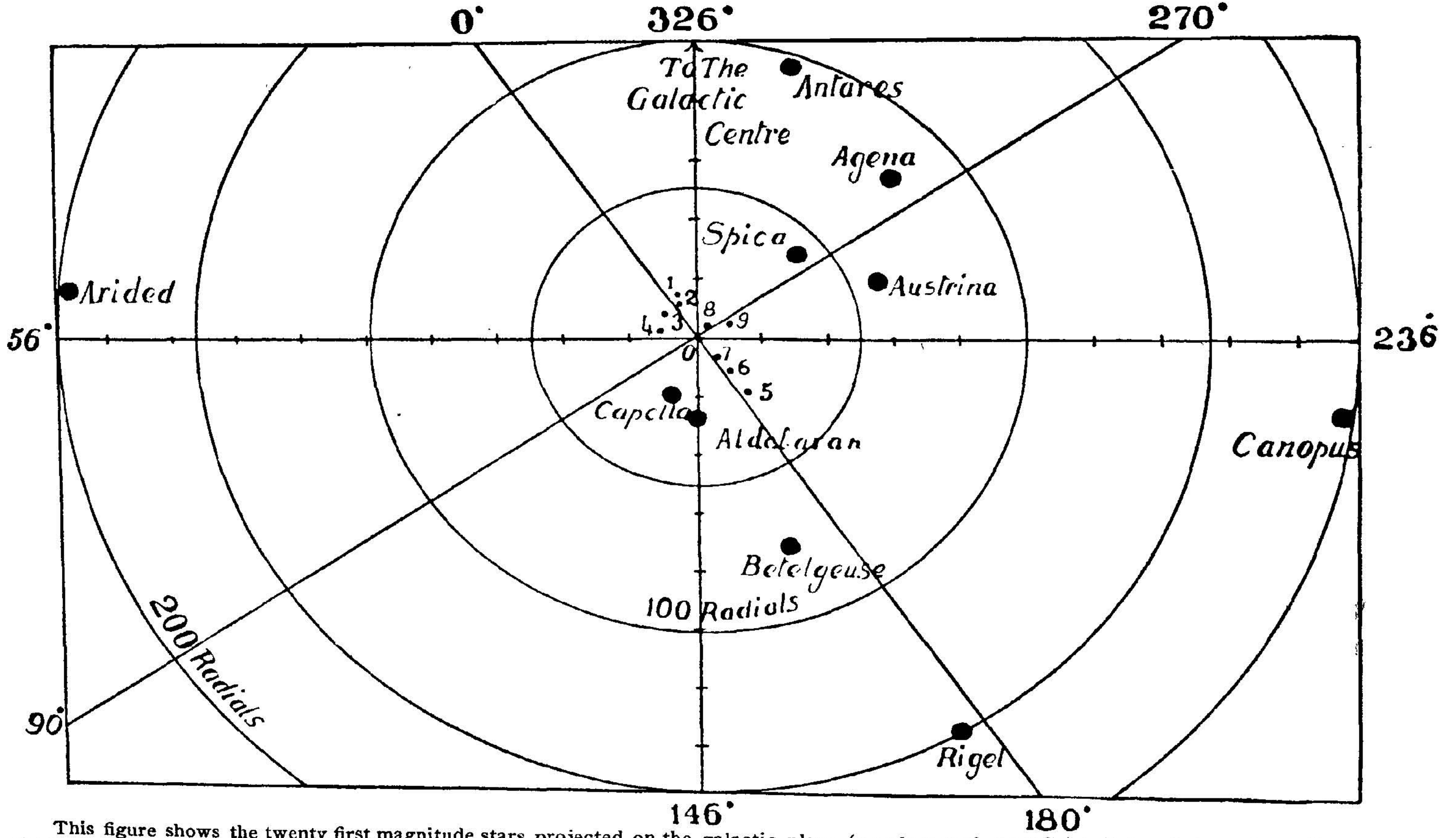
distance in radials, as given in col. 7, the reciprocal of the parallax, and taken from the Yale 1935 catalogue of parallaxes, amended by later revisions. The next two columns give respectively the distance R resolved perpendicular to (R sin β) and along the galactic plane (R cos β) in the direction λ. These values appear in columns 8 and 9. Figure 1 illustrates the result, the stars being shown according to their absolute magnitudes projected (λ, R cos β) on the galactic plane i.e. as seen from remote depths at the north galactic pole. Their λ, β positions are then projected on to the vertical plane passing through the galactic poles and along the line λ 56° to 236°, being normal to our sight line from λ 326°; such projection is expressed by the abcissa $x = R \cos \beta \sin (146° - \lambda)$ and ordinate $y = R \sin \beta$. This direct relation is sufficiently accurate considering the enormous distance of 10,000 radials. So we derive the grouping of our stars from this viewpoint and illustrate the same in figures 2 and 3. The projections are reduced by the fraction radials 3438′/10000 to read minutes of arc as undernoted along the edges and measured from the Sun's position. This position, is itself 33 radials north of the galactic plane which is therefore indicated by the broken horizontal line 11′ arc below the solar origin of the coordinates in fig. 2. Column 10 gives the distance (R′) of each star from the galactic centre view point and is nearly represented according to the stars distance by R cos β cos (146° − λ). The actual distance is however represented by $R' = 10{,}000 + R \cos \beta \cos (146° - \lambda)$. The final column gives the apparent magnitude (M′) from the given point taken to be 10,000 radials throughout, since R′ does not differ effectively from this figure; whence $M' = M + 15$ in all cases.

The figures illustrate the entire break up in the familiar groups for our first magnitude stars and in their brightness. Seen from the galactic centre these stars present a horse shoe cluster lying within an area equal to the lunar disc, gathered closely by its faintest member our Sun (of 20th mag.)—which clustering is what we would expect of the stars apparently brightest through greater nearness. Outside this clustering lie three notable exceptions Canopus, Rigel and Arided, flanking in a line somewhat inclined to the galactic plane.

In conclusion it should be pointed out that these our first magnitude stars are by no means to be recognised as such, seeing that the countless other stars (mainly invisible to us) in between, would fill up the foreground of our new viewpoint, not to mention the many stars near the galactic centre that would constitute the first magnitude array for any eyes there to see them.

Star	Constellation	λ	β	m	M	R	R sin β	R cos β	R′	M′
Sun	——	—	—	- 26·72	+4·9	0	0	0	10,000	19·9
Sirius	α Canis Majoris	194°·0	-9°·0	- 1·58	1·3	2·7	- 0·4	2·6	10,001	16·3
Canopus	α Carinae	228·2	-24·8	- 0·86	-7·6	220	-93	200	10027	7·4
Lugal	α Centauri	282·6	0·0	+ 0·06	4·4	1·3	0	1·3	9999	19·5
Vega	α Lyrae	34·4	19·0	0·14	0·6	8·2	2·7	7·7	9996	15·6
Capella	α Aurigae	129·3	3·6	0·21	-0·7	15·4	1·0	15·4	10014	14·3
Arcturus	α Bootis	343·6	69·7	0·24	-0·1	11·7	11·0	4·1	9996	14·9
Rigel	β Orionis	176·2	-25·7	0·34	-5·7	167	-72	150	10130	9·3
Procyon	β Canis Minoris	180·2	12·8	0·48	2·9	3·3	0·8	3	10002	17·9
Achernar	α Eridani	258·6	-58·2	0·60	-1·1	22	-19	11·7	9995	13·9
Betelgeuse	α Orionis	166·6	- 9·3	0·70	-3·9	83	-13	82	10077	11·1
Agena	β Centauri	278·5	2·0	0·86	-3·7	83	+ 2·9	83	9943	11·3
Altair	α Aquilae	14·5	- 8·9	0·89	2·4	4·9	- 0·8	4·9	9997	17·5
Astrina	α Crucis	257·0	0·2	1·03	-3·1	67	+ 0·2	67	9976	11·9
Aldebaran	α Tauri	147·8	-20·9	1·06	-0·4	20	- 7·2	18·8	10019	14·6
Pollux	β Geminorum	158·6	22·8	1·21	1·2	10	4·0	9·2	10009	16·3
Spica	α Verginis	282·7	51·6	1·21	-3·1	71	55	44	9968	11·9
Antares	α Scorpii	318·3	15·8	1·22	-3·8	100	27	97	9903	11·2
Fomalhaut	α Piscis Australis	345·5	-64·4	1·29	1·5	7·7	- 7·0	3·3	9997	16·9
Arided	α Cygni	51·0	1·7	1·33	5·2	200	6	200	9983	9·8
Regulus	α Leonis	192·2	48·8	1·34	-0·4	22	16·7	14·7	10009	14·6

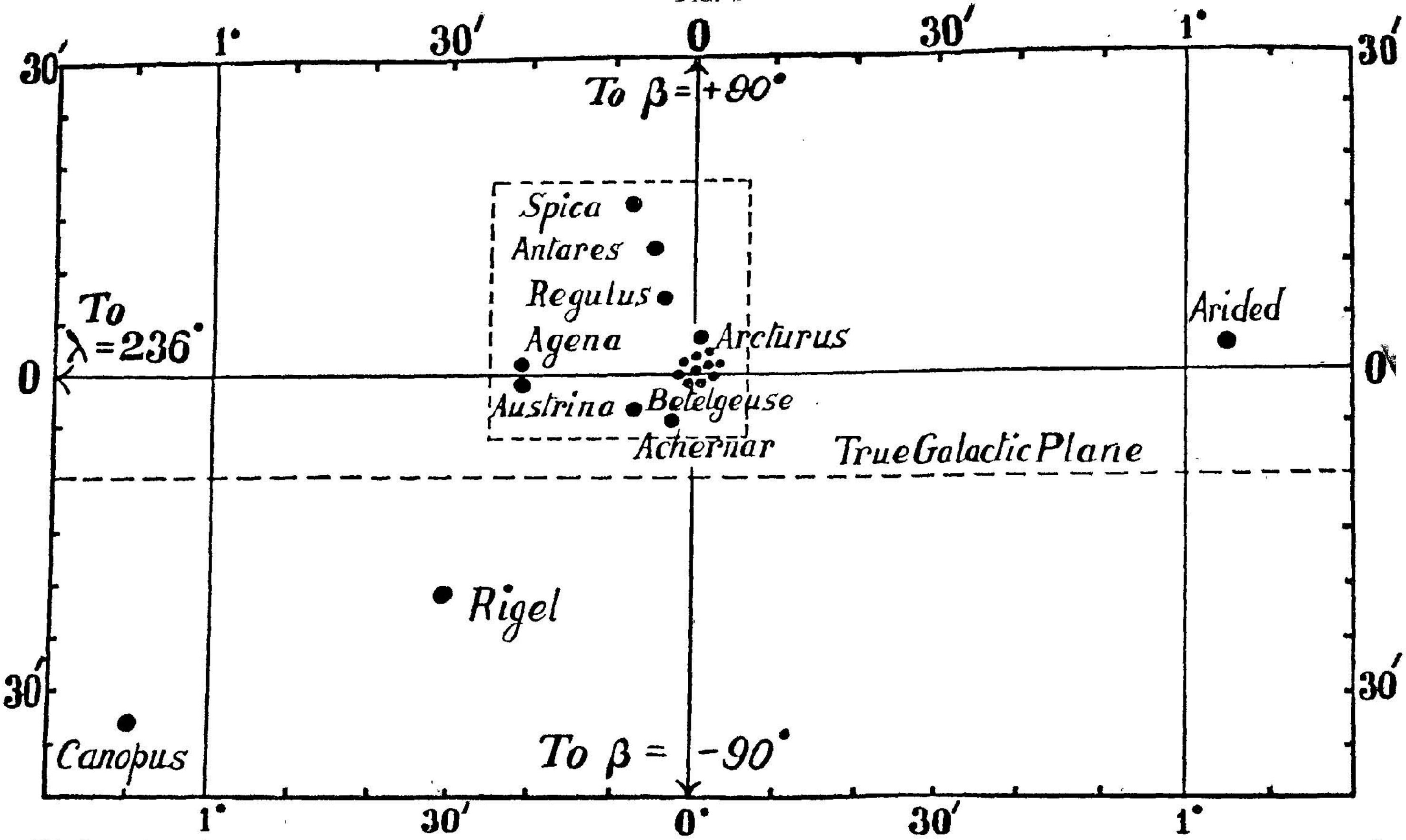
FIG. 1



This figure shows the twenty first magnitude stars projected on the galactic plane (*n.* pole α 192°, $\delta+27°$) i.e. R cos β along the λ direction from the Sun as centre. The concentric circles define the distances at 50 radials (163 light years) intervals. The actual distances of the stars are greater by the multiplier sec β. The magnitudes shown are absolute i.e. $M=m+5\text{-}5 \log R$. The stars marked 1 to 9 in the inner circle are:

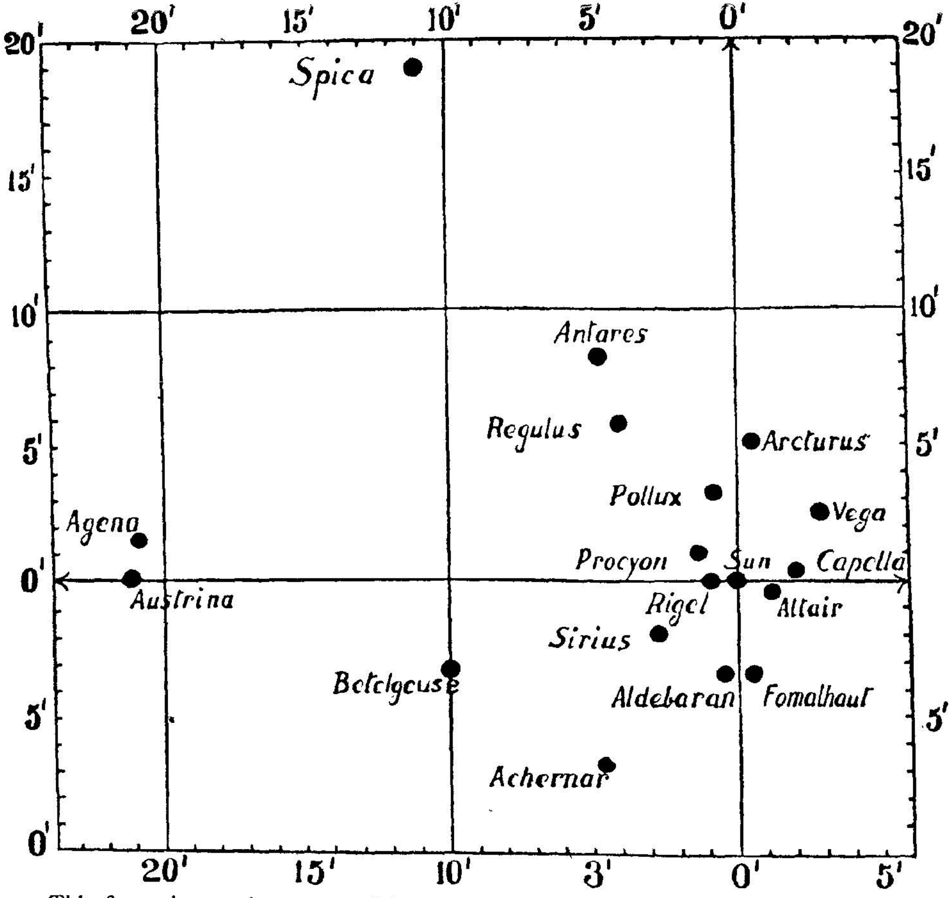
1. Arcturus 2. Fomalhaut 3. Altair 4. Vega 5. Regulus 6. Pollux 7. Sirius 8. Rigel 9. Achernar

FIG. 2



This figure shows the twenty first magnitude stars projected on to the vertical plane passing through the sun and galactic poles in the direction λ 56°–236° viewed from λ 326°, β 0° at 10,000 radials and towards λ 146°, β 0°. The scale in minutes of arc (originating from the sun) appears along the margin. The dotted line marks the real galactic plane – 33 radials below the sun or 11′ at 10,000 radials. The dotted square defines the limits enclosing seventeen first magnitude stars. The magnitudes shown are M′ = M + 15.

FIG 3



This figure is an enlargement of the central part of 30′ square in fig. 2 enclosing all but three of the twenty first magnitude stars, Canopus, Rigel and Arided. The magnitudes shown are as in fig. 2.

THE TRIANGULAR DISTRIBUTION

BY

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1. If we regard the Pearson Frequency curves from a naive point of view as glorified triangles, it becomes worth while to consider the properties of the triangular distribution, *per se.* We shall see, presently, that this distribution, the simplest that we know, has certain curious properties, which it does not share with any of the standard distributions. Our interest in this case has therefore some justification.

2. ABC is a triangle with BC horizontal, and containing X the middle point, D the foot of the perpendicular from A, E the foot of another perpendicular EF which bisects the area of the triangle and meets AC in F (DC>BD, without loss of generality), and M the orthogonal projection of the centroid G on BC.

We may now suppose that BAC represents a triangular distribution of limited range, for which D, E, and M mark the positions of the mode, the median, and the mean respectively. If EF cuts AX at K, we have AE parallel to XF and $CE^2 = CD \cdot CX$, so that

$$\frac{AK}{KX} = \frac{AE}{XF} = \frac{CE}{CX} = \frac{CD}{CE} = \sqrt{\frac{CD}{CX}} = t \text{ (say).}$$

$$\text{Since } 1 < \frac{CD}{CX} < 2, \text{ it follows that } 1 < t < \sqrt{2} < \frac{AG}{GX} \; (=2) \qquad \ldots(1)$$

Hence K lies between A and G, and E between D and M, i.e. the median lies between the mode and the mean, which is, of course, as it ought to be.

But, if we consider the ratio, say $R = \dfrac{\text{Mean} - \text{Mode}}{\text{Mean} - \text{Median}}$ we see that

$$R = \frac{AG}{KG} = \frac{AK + KX}{KX - \frac{1}{2}AK} = \frac{t+1}{1 - \frac{1}{2}t}$$

and, therefore, R lies between 4 and $4 + 3\sqrt{2}$, ...(2)
by virtue of (1).

This last result goes directly against the well known empirical value 3 of R, which is known to be approximately true for certain types of distributions.†

† *Curr. Sci.* Vol 9, no. 9. (1940), p. 411.

Further, it is interesting to note that the two extreme values 4 and $4+3\sqrt{2}$ occur under rather paradoxical circumstances. If the triangle ABC is isosceles, so that D and X coincide, we have R indeterminate, but $t=1$, which gives the limiting value 4 for R.

If the triangle ABC is right-angled at B, we have CD = CB = 2CX and $t=\sqrt{2}$; the triangular distribution reduces to a linear distribution, a sort of a limiting form of the J distribution, and R takes the maximum value $4+3\sqrt{2}$. The same value of R also corresponds to the case where C goes to infinity and the distribution becomes a combination of an oblique linear distribution and an infinite rectangular distribution, for

$$\underset{C\to\infty}{\mathrm{Lt}} \frac{DC}{XC} = \underset{DC\to\infty}{\mathrm{Lt}} \frac{2DC}{BD+DC} = 2.$$

3. Moments and Semi-invariants:

Let $DC=x_1$, $BD=x_2$, and $\Delta=\frac{1}{2}\,p\,(x_1+x_2)$; μ_r, μ'_r the r-th moments about the mean and the point D respectively.

$$\text{Then } \Delta\mu'_r = \frac{p}{x_1}\int_0^{x_1} x^r\,(x_1-x)\,dx+(-1)^r\,\frac{p}{x_2}\int_0^{x_2} x^r\,(x_2-x)\,dx$$

$$= \frac{p}{(r+1)(r+2)}\,\{x_1^{r+1}+(-1)^r\,x_2^{r+1}\}; \qquad \ldots (3)$$

$$\mu'_1 = \frac{x_1-x_2}{3};\quad \mu'_0=1 \qquad \ldots (4)$$

$$\text{Hence, } \Delta\mu_r = \Delta(\mu'_r-{}^rc_1\,\mu'_{r-1}\,\mu'_1+{}^rc_2\,\mu'_{r-2}\,\mu'^2_1-\cdots+(-1)^r\,\mu'_0\,\mu'^r_1)$$

$$= \frac{p}{(r+1)(r+2)}\,\{f(x_1)-f(-x_2)\},$$

$$\text{where } f(x) = x^{r+1}-{}^{r+2}c_1\,x^r\mu'_1+{}^{r+2}c_2\,x^{r-1}\,\mu'^2_1-\cdots+(-1)^r\,{}^{r+2}c_r\,\mu'^r_1\,x$$

$$= \frac{1}{x}\,\{(x-\mu'_1)^{r+2}+(-1)^r\,(r+2)\,x\mu'^{r+1}_1+(-1)^{r+1}\,\mu'^{r+2}_1\}$$

On further simplification,

$$\mu_r = \frac{2}{(x_1+x_2)(r+1)(r+2)}\left\{\frac{(x_1-\mu'_1)^{r+2}}{x_1}+(-1)^{r+2}\frac{(x_2+\mu'_1)^{r+2}}{x_2}\right\}$$

$$+\frac{2\,(-1)^{r+1}}{x_1\,x_2\,(r+1)(r+2)}\,\mu'^{r+2}_1; \qquad \ldots (5)$$

in particular, $\mu_2=\frac{1}{18}\,(x_1^2+x_1x_2+x_2^2)$, whence the standard deviation is $\sqrt{x_1^2+x_1x_2+x_2^2}/3\sqrt{2}$. ... (6)

When $x_1=x_2$, we have the symmetrical isosceles-triangle distribution for which

$$\mu_{2r+1}=0,\ \mu_{2r}=\frac{x_1^{2r}}{(r+1)(2r+1)}; \qquad \ldots (7)$$

so that

$$\mu_2 = \frac{x_1^2}{6},\ \mu_4 = \frac{x_1^4}{15},\ \beta_2 = \frac{\mu_4}{\mu_2^2} = 2{\cdot}4 < 3, \qquad \ldots \quad (8)$$

which show that all symmetrical isosceles-triangle distributions are *platy-kurtic* (flat-topped)!

The semi-invariants K_r† are given by the identity

$$\sum_2^\infty K_r \frac{t^r}{r!} \equiv \log \left\{ 1 + \sum_2^\infty \mu_r \frac{t^r}{r!} \right\} \qquad \ldots \quad (9)$$

Substituting for μ_r from (7) in (9), the right side of (9) takes the form

$$\log \left\{ 1 + 2 \sum_{r=1}^{\infty} \frac{t^{2r}\ x_1^{2r}}{(2r+2)!} \right\}$$

i.e. $\log \left\{ \dfrac{2}{x_1^2\, t^2} \cdot \dfrac{e^{x_1 t} + e^{-x_1 t} - 2}{2} \right\}$

i.e. $2 \log \left(2 \sinh \dfrac{x_1 t}{2} \right) - 2 \log x_1 t$

Differentiating both sides of (9) with respect to 't' and equating coefficients of powers of t, we get $K_{2r+1} = 0$, and
‡ $K_{2r} = (-1)^{r-1}\ B_r x_1^{2r}/r$, where B_r is the r-th Bernoulli-number ... (10)

In particular, $K_2 = B_1\, x_1^2 = \dfrac{x_1^2}{6},\ K_4 = -\dfrac{1}{2}\ B_2\, x_1^4 = -\dfrac{x_1^4}{60}$, and so on.

Thus, semi-invariants of odd order vanish, and those of the even order are positive or negative according as the order is singly even or doubly even.

It is rather curious that Bernoulli's numbers should be associated with the semi-invariants of an isosceles triangle distribution.

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† E. A. Cornish and R. A. Fisher: Moments and cumulants in the specification of distributions, Revue de l'Institut International de Statisque, 5 Annee-Liv 4. Jan 1938.

‡ The differential coefficient w.r.t. 't' of the right side of (9) is $x_1 \coth \dfrac{tx_1}{2} - \dfrac{2}{t}$.

But $\coth x = 1 + \dfrac{2}{e^{2x} - 1} = \dfrac{1}{x} + \sum\limits_1^\infty (-1)^{r-1}\ B_r\ 2^{2r} \cdot \dfrac{x^{2r-1}}{(2r)!}$ where B_1, B_2, B_3, ... are Bernoulli's numbers. Hence

$$x_1 \coth \frac{x_1 t}{2} - \frac{2}{t} = \sum_1^\infty (-1)^{r-1}\, 2B_r \cdot \frac{t^{2r-1} x_1^{2r}}{(2r)!} = \sum_2^\infty K_r \cdot \frac{t^{r-1}}{(r-1)!}$$

NOTES AND DISCUSSIONS

A Property of Pascal's Triangle

1. The object of this note is to draw attention to a curious property of the binomial coefficients written out as in Pascal's Arithmetical Triangle.

Write down the binomial coefficients in the order in which they occur in Pascal's triangle, but omitting the last number unity in each line:—

1	2	...	...	...	...	...	...	...
1	3	3	...	...	...	...	...	..
1	4	6	4	...	...	...	...	...
1	5	10	10	5	...	...	...	...
1	6	15	20	15	6	...	...	...
1	7	21	35	35	21	7	...	...
1	8	28	56	70	56	28	8	...
...	...	...	...	...	...	...	...	...

Then if nets are drawn as above so as to include an odd number of rows and the same number of columns ($n \geqslant 3$), the determinants so formed all vanish.

2. We define Bernoulli's numbers B_r by means of the equation

$$\frac{x}{e^x - 1} = B_0 + \frac{B_1}{1!}x + \frac{B_2}{2!}x^2 + \frac{B_3}{3!}x^3 + \cdots \quad \ldots \quad (1)$$

then it can be easily seen† that

$$B_0 = 1,\ B_2 = -\tfrac{1}{2};\ B_3 = B_5 = B_7 = \cdots = B_{2n+1} = \cdots = 0. \quad \ldots \quad (2)$$

and that

$$1 + \binom{n}{1} B_1 + \binom{n}{2} B_2 + \cdots + \binom{n}{n-1} B_{n-1} = 0 \quad \ldots \quad (3)$$

† Knopp: *Theory and Applications of Infinite Series* (1928). p. 183.

If in (3) we set $n=2, 3, 4, \cdots (r+1)$ in order and out of the r resulting equations eliminate $B_1, B_2, B_3, \cdots B_{r-1}$, we have

$$(-)^r (r+1)!\, B_r = \begin{vmatrix} 1 & 2 & 0 & 0 & \cdots & 0 \\ 1 & 3 & 3 & 0 & \cdots & 0 \\ 1 & 4 & 6 & 4 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & \binom{r+1}{1} & \binom{r+1}{2} & \cdot & \cdots & \binom{r+1}{r-1} \end{vmatrix} \quad \ldots (4)$$

Hence, if r is odd, say $r=2n+1$ $(n>0)$, we have from (2) the required result.

3. It may be noted in passing that on division by $n!$, (3) becomes

$$\frac{1}{n!} + \frac{1}{(n-1)!}\frac{B_1}{1!} + \frac{1}{(n-2)!}\frac{B_2}{2!} + \cdots + \frac{1}{1!}\frac{B_{n-1}}{(n-1)!} = 0 \quad \ldots (5)$$

so that we find as before that $(-)^r \dfrac{B_r}{r!} = \Delta(r)$ where

$$\Delta(r) = \begin{vmatrix} \frac{1}{2!} & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \cdots & 0 \\ \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \frac{1}{(r+1)!} & \frac{1}{r!} & \frac{1}{(r-1)!} & \frac{1}{(r-2)!} & \cdots & \frac{1}{2!} \end{vmatrix} \quad \ldots (6)$$

Hence we find that $\Delta(r)=0$ whenever the determinant is of odd order.

Trivandrum. K. BALACHANDRAN.

A method of forming the equation of squared differences

Let

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

be an equation of the nth degree. Put $x+iy$ for x in $f(x)$, expand $(x+iy)^k$, $k=1, 2, \cdots, n$ and collect the terms not containing i and those containing i. Then we can write

$$f(x+iy) = g(x, y) + ih(x, y) \quad \ldots (1)$$

It must be noted that the right side of (1) is *not* the expression of $f(x+iy)$ into real and imaginary parts. This will be so only if $a_0, \ldots, a_n$ are real and x and y are real; this is not supposed to be the case. Changing i into $-i$ we get

$$f(x-iy)=g(x, y)-i\,h(x, y) \qquad \ldots (2)$$

It is evident that $g(x, y)$ contains only even powers of y and $h(x, y)$ only odd powers. Hence writing $u=y^2$, we get

$$g_1(x, u)=g(x, y); \quad h_1(x, u)=h(x, y)/y \qquad \ldots (3)$$

where g_1 and h_1 are polynomials in x and u. Let α and β be any two roots of $f(x)=0$. Then we can choose x and y so that

$$x+iy=\alpha; \; x-iy=\beta \qquad \ldots (4)$$

so that we get

$$x=\frac{\alpha+\beta}{2}; \quad y=\frac{\alpha-\beta}{2i} \qquad \ldots (5)$$

while $g(x, y)=h(x, y)=0$ in virtue of (1) and (2) and so if we eliminate u between the expressions in (3) equated to zero, the resulting equation

$$f_2(x)=0, \qquad \ldots (6)$$

is satisfied by $x=\frac{1}{2}(\alpha+\beta)$. Similarly if we eliminate x between the equations (3), the resulting equation

$$f_3(u)=0, \qquad \ldots (7)$$

is satisfied by $u=-(\alpha-\beta)^2/4$. If the degree of the eliminants in (6) and (7) are each ${}_nC_2$ then we get the equations whose roots are $\frac{1}{2}(\alpha+\beta)$, $\frac{1}{2}(\alpha+\gamma)$, . . etc. and $-\frac{1}{4}(\alpha-\beta)^2, -\frac{1}{4}(\alpha-\gamma)^2$, . . etc. If the degrees of (6) and (7) are greater than ${}_nC_2$ they contain respectively as factors the equations whose roots are $\frac{1}{2}\times$ the sum of any two roots and $-\frac{1}{4}\times$ (the square of the difference of any roots) of the equation $f(x)=0$.

As an example, let us take the equation

$$x^5+qx+r=0.$$

Here

$$g(x, y) = x^5-10x^3y^2+5xy^4+qx+r$$

$$\frac{h(x, y)}{y} = 5x^4+y^4-10x^2y^2+q$$

so that

$$g_1(x, u)=x^5-10x^3u+5xu^2+qx+r=0$$

$$h_1(x, u)=5x^4+u^2-10x^2u+q=0$$

Multiplying g_1 by $5x$ and subtracting from h_1 we easily get

$$u=(24x^4+4qx-r)/40x^3$$

The eliminant

$$f_2(x) = 8000\ x^{10} + (24x^4 + 4qx - r)^2$$
$$- 400\ x^5\ (24x^4 + 4qx - r) + 1600qx^6 = 0$$

is of the 10th degree and ${}_5C_2 = 10$. Hence $f_2(x) = 0$ is the equation whose roots are the average of any two roots of the given equation.

From $g_1(x, u)$ we can solve for x^2 and thus eliminate u. The resulting equation is

$$5^3\ [16\ (4\ u^2 - q)\ (8\ u^3 - uq) - 5\ r^2]^2 = 16^2\ (4\ u^2 - q)^3\ (44\ u^2 - q)^2$$

which is also of the 10th degree and hence is the equation of the squared differences. The value of $\Pi\ (\alpha - \beta)^2$ for the given equation is given by

$$\Pi(\alpha - \beta)^2 = \frac{4^{10}\ \{5^5\ r^4 + 16^2\ q^5\}}{5^3 \times 16^2 \times 4^2 \times 8^2 - 16^2 \times 4^3 \times 44^2}$$
$$= 4^2\ (5^5\ r^4 + 16^2\ q^5)/2479$$

The condition for equal roots is

$$5^5\ r^4 + 16^2\ q^5 = 0$$

and this can be verified by eliminating x between

$$x^5 + qx + r = 0 \text{ and } 5x^4 + q = 0.$$

Kapurthala. AMRITSAGAR PURI.

A property of the Tetrahedron

Let a tetrahedron be determined by the four points A, B, C, D. We shall denote the angle between the planes passing through the side BC i.e. the angle between the planes ABC and BCD by the symbol $\angle$ BC.

Using similar symbols for the other such angles we shall prove,

$$\frac{BC \cdot AD}{\sin \angle BC \cdot \sin \angle AD} = \frac{CA \cdot BD}{\sin \angle CA \cdot \sin \angle BD} = \frac{AB \cdot CD}{\sin \angle AB \cdot \sin \angle CD}$$

Proof:

Let P be the distance of D from the plane ABC. Then if p_1, p_2, p_3 be the respective lengths of the perpendiculars from D on the lines BC, CA, AB, we get

$$P = p_1 \sin \angle BC = p_2 \sin \angle CA = p_3 \sin \angle AB.$$

Let α, β, γ, δ respectively be the areas of the triangles BCD, CAD, ABD, ABC; and V the volume of the tetrahedron. Then multiplying the above equality with δ,

$$3V = \frac{2\ \delta\alpha \sin \angle BC}{BC} = \frac{2\ \delta\beta \sin \angle CA}{CA} = \frac{2\ \delta\gamma \sin \angle AB}{AB}$$

Similarly we get

$$3V = \frac{2\,\beta\gamma \sin \angle AD}{AD} = \frac{2\,\alpha\gamma \sin \angle BD}{BD} = \frac{2\,\alpha\beta \sin \angle CD}{CD}$$

$$\therefore \quad \frac{9\,V^2}{4\,\alpha\beta\gamma\delta} = \frac{\sin \angle BC \cdot \sin \angle AD}{BC \cdot AD} = \frac{\sin \angle CA \cdot \sin \angle BD}{CA \cdot BD}.$$

Hence the result follows.

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An expression for Jordan function $\varphi_k(N)$ in terms of Euler function $\varphi(N)$

If $\varphi_k(N)$ denotes the Jordan function representing the number of sets of k numbers not greater than N whose greatest common divisor is prime to N, and $\varphi(N)[=\varphi_1(N)]$, the Euler function representing the number of numbers not greater than N and prime to N, then

$$\varphi_k(N) = \sum_{d_1 d_2 \ldots d_k = N} \varphi(d_1)\,\varphi(d_2^2)\,\varphi(d_3^3) \cdots \varphi(d_k^k).$$

Proof

$$\sum_{N=1}^{\infty} \frac{\varphi_k(N)}{N^s} = \frac{\zeta(s-k)}{\zeta(s)}, \qquad s > k+1$$

where $\zeta(s)$ denotes the function $\sum_{N=1}^{\infty} 1/N^s$

$$\text{Also } \sum \frac{\varphi(N^r)}{N^s} = \sum \frac{N^{r-1}\varphi(N)}{N^s} = \sum \frac{\varphi(N)}{N^{s-r+1}} = \frac{\zeta(s-r)}{\zeta(s-r+1)}.$$

Therefore,

$$\prod_{r=1}^{k} \sum \frac{\varphi(N^r)}{N^s} = \prod_{r=1}^{k} \frac{\zeta(s-r)}{\zeta(s-r+1)} = \frac{\zeta(s-k)}{\zeta(s)} = \sum \frac{\varphi_k(N)}{N^s};$$

whence, by Dirichlet multiplication we get

$$\varphi_k(N) = \sum_{d_1 d_2 \cdots d_k = N} \varphi(d_1)\,\varphi(d_2^2) \cdots \varphi(d_k^k),$$

Annamalainagar.

P. KESAVA MENON.

SOLUTIONS TO QUESTIONS

Question 1744

(A. A. KRISHNASWAMY AYYANGAR):—If a/b, c/d be two positive fractions such that $ad-bc>1$, there exists at least one intermediate fraction with denominator smaller than b or d whichever is greater.

Solution by V. Rangachariar

The proof is based on the fact that if p/q is a positive fraction, there exists another fraction x/y where $x<p$ and $y<q$ such that we can make $py-qx=1$ or -1 as we choose. For if p/q be converted into a continued fraction and p'/q' be the penultimate convergent, we have $p'<p$ and $q'<q$ and $pq'-p'q=\pm1$. Let $r=p-p'$ and $s=q-q'$, Then $ps-qr=\mp1$. Thus taking x, y respectively equal to p', q' or r, s we make $py-qx=+1$ or -1 as we choose.

To prove the proposition let us first suppose $b>d$. Let α/β be found ($\alpha<a$, $\beta<b$) so that $a\beta-b\alpha=1$. Evidently α/β is distinct from c/d. And as c/d being a fraction of smaller denominator cannot lie between a/b and α/β we have $a/b>\alpha/\beta>c/d$ where $\beta<b$. If $d>b$ let γ/δ be found so that $c\delta-\gamma d=-1$ and $\gamma<c$, $\delta<d$. We have $\gamma d-c\delta=1$. As a/b cannot lie between γ/δ and c/d we have $a/b>\gamma/\delta>c/d$ where $\delta<d$.

Question 1780

(M. S. SRINIVASAN):—If p be prime and $M=1.\ 3.\ 5.\ 7.\ .\ .\ (p-2)$, prove that $M^4\equiv1 \pmod p$.

Solution by Hansraj Gupta.

We have $M = 1.\ 3.\ 5.\ 7.\ .\ .\ (p-2)$,

$$\equiv (-1)^{(p-1)/2}.\ (p-1)(p-3)(p-5)\ .\ .\ .\ 2 \pmod p.$$

Therefore $M^2 \equiv (-1)^{(p-1)/2}.\ (p-1)! \pmod p$

$$\equiv (-1)^{(p+1)/2}. \pmod p.$$

Hence $M^4 \equiv 1 \pmod p$.

Moreover $M^4 \equiv 1 \pmod{16}$

Therefore $M^4 \equiv 1 \pmod{16p}$.

Also solved by Messrs. P. Kesava Menon, Madhusudan Prasad, K. Balachandran, C. S. Venkataraman and the proposer.

Question 1794

(P. KESAVA MENON):—Find the value of

$$\sqrt[3]{\left[\frac{3}{2}+\frac{1\times 3}{2^2}\sqrt[3]{\left\{\frac{5}{2}+\frac{3\times 5}{2^2}\sqrt[3]{\left(\frac{7}{2}+\frac{5\times 7}{2^2}\times\cdots\text{ to }\infty\right)}\right\}}\right]}$$

Solution by K. Balachandran.

Let $f(n)=(2n+1)/2$. Then it is obvious that

$$f(n)=\sqrt[3]{\frac{2n+1}{2}+\frac{(2n-1)(2n+1)}{2^2}f(n+1)}$$

so that

$$f(n)=\sqrt[3]{\left[\frac{2n+1}{2}+\frac{(2n-1)(2n+1)}{2^2}\sqrt[3]{\left\{\frac{2n+3}{2}+\frac{(2n+1)(2n+3)}{2^2}\times\sqrt[3]{\left(\cdots\text{ to }\infty\right)}\right\}}\right]}$$

Putting in this $n=1$, we have the required value as $f(1)=\frac{3}{2}$

Also it is obvious from the above that if we start with any other radical in (1) the value of the infinite expression is given by the first term under the radical: thus, for example,

$$\sqrt[3]{\left[\frac{5}{2}+\frac{3\cdot 5}{2^2}\sqrt[3]{\left\{\frac{7}{2}+\frac{5\cdot 7}{2^2}\sqrt[3]{\left(\frac{9}{2}+\frac{7\cdot 9}{2^2}\times\cdots\text{ to }\infty\right)}\right\}}\right]}=\frac{5}{2}.$$

For two problems quite similar to (1) see Question 289 (i) and (ii), page 323, "*Collected Papers of Srinivasa Ramanujan*".

Also solved by Messrs. S. Venketanarsimhan, C. R. Krishnamurthi, and by the proposer.

[None of the solvers has considered the convergence of this sequence of operations. Without such a discussion the solution is incomplete—Ed.]

Question 1795

(K. BALACHANDRAN):—If

$$\Delta(r)=\begin{vmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \cdots & 0 \\ \frac{1}{5!} & \frac{1}{4!} & \frac{1}{2!} & 1 & \cdots & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \frac{1}{(2r-1)!} & \frac{1}{(2r-2)!} & \frac{1}{(2r-4)!} & \cdot & \cdots & \frac{1}{2!} \end{vmatrix}$$

and $r>1$, show that $(2r-1)!\ \Delta(r)$ is an integer.

Solution by the Proposer

If the rows of $\Delta(r)$ are respectively multiplied by $1!, 3!, \cdots (2r-1)!$ and the columns are divided respectively by $1, 1!, 3!, \cdots, (2r-3)!$, the result is $(2r-1)!\ \Delta(r)$ while each element of the new determinant is a binomial co-efficient or zero. Hence $(2r-1)!\ \Delta(r)$ is an integer.

Remark. It can be shown from the identities $\sin z = \tan z \cos z$ and the expansion of $\tan z$ in a power series whose coefficients involve Bernoulli's numbers, that

$$(-1)^{r-1}\, 2^{2r}\, (2^{2r}-1)\, \frac{B_{2r}}{2r} = (2r-1)!\ \Delta(r)$$

so that we get a proof of the known result* that $2^{2r}(2^{2r}-1)\,\frac{B_{2r}}{2r}$ is an integer.

Also solved by V. P. Akhaury and G. B. Banerjee.

Question 17232

(*Educational Times*)

(M. T. NARANIENGAR) :—If the normal at P meets the parabola in Q, and the normal at Q the axis in G, show that APQG is cyclic.

Solution by V. A. Mahalingam

Let A be the vertex of the parabola $y^2 = 4ax$, P the point "t" $(at^2, 2at)$ and Q the point $t' = -t - \frac{2}{t}$ where the normal at P cuts the curve. Let the circle APQ cut the parabola again at R and let AR be the line $y = mx$. Then the equation of any conic through A, P, Q, R will be of the form

$$y^2 - 4ax + k\,(y - mx)\,(y + xt - 2at - at^3) = 0.$$

Using the conditions for a circle, this equation simplifies into

$$(1+t^2)\,(y^2 - 4ax) - (y - tx)\,(y + tx - 2at - at^3) = 0$$

This circle cuts the x-axis at a distance $a(t^4+6t^2+4)/t^2$ from A.

The normal at t' cuts the axis at a distance x given by

$xt' = 2at' + at'^3$ i.e. $x = 2a + at'^2$ which, with $t' = -t - \frac{2}{t}$, becomes $a(t^4+6t^2+4)/t^2$. Hence the circle APQ passes through G.

Also solved by G. B. Banerji, K. Balachandran, Mohd. Khwaja Mohiuddin, B. Natarajan, M. A. Sundararajan, D. Venkataraman and C. S. Venkataraman.

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* Nielson, *Traite de Nombres de Bernoulli.*

QUESTIONS FOR SOLUTION

1802. (AMRITSAGAR PURI):—Let (nx) denote the fractional part of nx and

$$p(nx) = \begin{cases} \frac{1}{2} - (nx), & \text{if } nx \text{ is not an integer;} \\ 0, & \text{if } nx \text{ is an integer.} \end{cases}$$

Show that the infinite series

$$f(x) = \frac{p(x)}{1} + \frac{p(2x)}{2} + \frac{p(3x)}{3} + \cdots\cdots$$

converges for all positive rational values of x and that, given $A > 0$ however large, it is possible to find a rational number x such that $f(x) > A$.

1803. (K. BALACHANDRAN):—Show that

$$\frac{x^{2n+2}}{2n+2} + \frac{1}{2}\,\frac{x^{2n+4}}{2n+4} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^{2n+6}}{2n+6} + \cdots\cdots$$

$$= \frac{(-1)^n \sqrt{(1-x^2)}}{3\cdot 5 \cdots (2n+1)} \begin{vmatrix} x^2+2 & 3 & 0 & 0 & \cdot & \cdot & 0 \\ x^4 & -4 & 5 & 0 & \cdot & \cdot & 0 \\ x^6 & 0 & -6 & 7 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x^{2n} & 0 & 0 & 0 & \cdot & \cdot & -2n \end{vmatrix}$$

Similarly express the series

$$\frac{x^{2n+1}}{2n+1} + \frac{1}{2}\,\frac{x^{2n+3}}{2n+3} + \frac{1\cdot 3}{2\cdot 4}\,\frac{x^{2n+5}}{2n+5} + \cdots\cdots$$

as a determinant

1804. (P. KESAVA MENON):—If S_n denote the sum of the nth powers of the roots of the equation $x^m + ax + b = 0$, show that

$$S_m{}^2 = (-1)^m \{mb^m - m^2 a^m b\}$$

$$S_{mr} = (-1)^r\, b^r m \quad \text{if } r < m-1$$

$$S_{m(m-1)} = (-1)^{m-1} \{mb^{m-1} - (m-1)\, a^m\}$$